Flexoelectricity and Thermal Fluctuations of Lipid Bilayer Membranes: Renormalization of Flexoelectric, Dielectric and Elastic Properties

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Abstract

Thermal fluctuations renormalize the bending elasticity of lipid bilayers. This well-studied effect is a cornerstone in the study of several membrane biophysical phenomena. Analogously, nearly all membranes are endowed with an *electromechanical* coupling called flexoelectricity that admits membrane polarization due to curvature changes. Flexoelectricity is found to be important in a number of biological functions including hearing, ion transport and in some situations where mechanotransduction is necessary. Very little is known about the interplay between thermal fluctuations and flexoelectricity. In this work, we explore how the apparent flexoelectricity is altered due to thermal fluctuations, and further, how the elastic and dielectric properties are renormalized due to flexoelectricity. We find that the apparent bending rigidity is *softened* by flexoelectricity and discuss the ramifications for interpreting existing experimental work.

1 Introduction

Lipid bilayer membranes are microscopically quite complex–yet, their behavior over large length scales is governed by just a few continuum parameters, e.g., bending modulus and surface tension govern shape changes and spectrum of thermal fluctuations [3]. The bending modulus, for instance, sets the scale for the energy cost of out of plane fluctuations. For most lipid-bilayers, this is between 10-20 $k_B T$ [4]. As well articulated by Deserno [5], "it is big enough such that the bilayer will not fluctuate into pieces" due to thermal energy but simultaneously not too excessive; thus allowing small sources of bio-energy such as ATP molecules and others to deform the membranes [5, 4].

In the classical continuum setting, for a given out of plane displacement h, the curvature-elastic energy is given by [3]:

$$H[h] = \int_{\mathbb{S}} \left[\frac{1}{2} \kappa_b (K_h)^2 \right] J_h, \tag{1}$$

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where J_h is the Jacobian measuring the area of the deformed membrane, K_h is the total mean curvature and κ_b is the bending modulus. Since the early and pioneering work of [3], it is now well-appreciated that thermal fluctuations *soften* the renormalized bending modulus. Barring a controversial work [6] which predicted *enhancement* in the rigidity of the renormalized bending modulus, most other works (analytical or numerical) predict similar qualitative behavior: $\kappa_b^R = \kappa_b - \frac{\alpha}{4\pi} k_B T \ln \frac{L}{t}$ [8, 9, 10, 11]. Here t is the thickness of the membrane and L is the characteristic length scale over which the coarse-graining is performed. The papers mentioned differ in their derivation of the constant α. Kleinert [12] resolved these discrepancies and conclusively showed that $\alpha = 3$ in addition to pointing out various subtleties related to the coarse-graining procedure. The reader is referred to the article by Morse and Milner [13] regarding the correct use of the integration measure while evaluating the partition function [see also Ref. [14] for an insightful tutorial]. The continuum formulation of the elastic response of membranes of which (1) is a subset, has been widely used to understand mechanical signaling, cell adhesion, steric repulsion between membranes, phase diagram of vesicles, protein-membrane interactions among others [cf [15] and references therein].

The focus of the current work is electromechanical coupling and specifically, *flexoelectricity*. Simply put, flexoelectricity is the appearance of a net dipole moment due to changes in curvature. Mathematically,

$$\mathbf{P} = f_e K_h \mathbf{n}. \tag{2}$$

Here $\mathbf{P}$ is the polarization (per unit area), $\mathbf{n}$ is the normal vector and f_e is the flexoelectric coupling constant that dictates the strength of the electromechanical coupling. While well-known in the liquid crystal literature, its phenomenology and microscopic underpinning for biological membranes were established by Petrov in a pioneering sequence of works—see [16, 17, 18, 19, 20, 21, 22] and references therein. The coupling constant f_e has been measured experimentally for a number of membranes and is generally found to be between: 0.3×10^{-19} to $151 \times 10^{-19} C$ depending on the charge conditions of lipids and the ionic environment [see Table1 in Ref. [19, 20]]. A value of $20 \times 10^{-19} C$ is a good representative value for illustrative calculations. Perhaps the most conclusive experimental evidence of flexoelectricity biological membranes (specifically the converse effect) was recently provided in a work by Brownell et. al. [28] where its role in cell-membrane tether formation was also elucidated. Indeed, past research suggests that flexoelectricity underlies an important coupling between mechanical deformation and electric stimulus which has ramifications for ion transport [17], hearing mechanism [23, 24, 25, 26, 27], tether formation [29, 28] among others. For example, Raphael et. al. [23] indicated that flexoelectricity is the major mechanism behind outer hair cell electromotility impacting cochlear amplification and sharp frequency discrimination. Despite a number of works on flexoelectricity, its ramification for biological phenomena is still *relatively* understudied.

In this paper, recognizing that flexoelectricity is a universal property of biological membranes, we propose a simple theoretical formulation of flexoelectricity and study the impact of thermal fluctuations on the renormalization of the flexoelectric coupling coefficient as well as the impact of flexoelectricity on other continuum properties: dielectric constant and bending modulus. We note here that other authors have recognized that flexoelectricity may renormalize mechanical properties of liquid crystals albeit the influence of thermal fluctuations was not the focus of those works e.g. [1]. Jewell [7] presents a nice exposition on the correspondence between liquid crystals and membrane mechanics while, specifically for the influence of flexoelectricity on membrane mechanical properties, the reader is referred to works by Helfrich [2] and Rey [35]

2 Theoretical Formulation

Consider a membrane of in-plane size L and thickness t. To account for flexoelectricity, the membrane is described by state variables $(\mathbf{P}, h)$, where $\mathbf{P}$ is the polarization (per unit area) and h is the out-of-plane displacement of the mid-plane of the membrane. Suppose that at the zero temperature, the membrane occupies the domain $\mathbb{S} := (0, L)^2$ on the xy-plane. Let

$$J_h = \sqrt{1 + |\nabla h|^2}, \quad \mathbf{n}_h = \frac{(-\nabla h, 1)}{J_h}, \quad K_h = \nabla \cdot \left[\frac{\nabla h}{J_h}\right], \quad G_h = \frac{\det(\nabla \nabla h)}{J_h^4},$$

where J_h is the Jacobian measuring the area of the deformed membrane relative to the flat reference $\mathbb{S}$, $\mathbf{n}_h$ is the unit normal vector on the membrane, and K_h (G_h) is the total (Gaussian) curvature, respectively. We propose the following Hamiltonian to describe elastic-flexoelectric membrane:

$$H[\mathbf{P}, h] = \int_{\mathbb{S}} \left[\frac{1}{2}a[(P_x)^2 + (P_y)^2] + \frac{1}{2}a_z(P_z)^2 + f\mathbf{P} \cdot \mathbf{n}_h K_h + \frac{1}{2}\kappa_b(K_h)^2 + \kappa_g G_h + \lambda\right] J_h, \qquad (3)$$

where $\frac{1}{2}a[(P_x)^2 + (P_y)^2] + \frac{1}{2}a_z(P_z)^2$ accounts for the polarization and associated nonlocal electric field energy. If the membrane is a linear dielectric with permittivity ϵ, we can show that (Appendix 1, Section 5) for polarization $\mathbf{P}$ varying slowly on $\mathbb{S}$ (i.e., the long wavelength limit),

$$a = \frac{1}{(\epsilon - \epsilon_0)t}, \qquad a_z = \left(\frac{1}{\epsilon - \epsilon_0} + \frac{1}{\epsilon_0}\right)\frac{1}{t}. \qquad (4)$$

As usual, $\frac{1}{2}\kappa_b(K_h)^2 + \kappa_g G_h$ is the Helfrich-Canham bending energy term and λ is the surface stress. Finally, the coupling term $f\mathbf{P} \cdot \mathbf{n}_h K_h$ arises from the flexoelectric effect that is the focus of the current analysis.

In the absence of external mechanical or electrical fields, we assume that the trivial state $(\mathbf{P}, h) = 0$ is stable. Then, by standard stability test we can show that the following constraints apply to the material properties [31]:

$$a > 0, \quad \kappa_b > 0, \quad \kappa_g < 0, \quad \kappa_b + \frac{\kappa_g}{2} > \frac{f^2}{a_z}. \qquad (5)$$

Before proceeding further, it is instructive to link the flexoelectricity coefficient appearing in our theoretical framework to the often used expression in the literature (2). In (2) the symbol $\mathbf{P}$ represents polarization per unit area with the dimension of C/m, and hence f_e has the dimension of C which is quantitatively different with flexoelectric coefficient in our expressions. In our theory, for a homogenous membrane and in the absence of external electric field, equation (3) implies that

$$P = -\frac{f}{a_z}K_h. \qquad (6)$$

Comparing with (2), we find

$$f_e = -\frac{f}{a_z}. \qquad (7)$$

Therefore our definition of flexoelectric constant f has the dimension of Nm/C.

3   Renormalization of the Membrane Properties due to Thermal Fluctuations

We remark that $a, a_z, f, \kappa_b, \kappa_g, \lambda$ in (3) are macroscopic (phenomenological) material constants of the flexoelectric membrane. Our goal is to find how these material constants depends on thermal fluctuations at a finite temperature. To this end, we decompose the state variables as

$$(\mathbf{P}, h) = (\bar{\mathbf{P}}, \bar{h}) + (\mathbf{P}', h'), \tag{8}$$

where $(\bar{\mathbf{P}}, \bar{h})$ is the average polarization and out-of-plane displacement and $(\mathbf{P}', h')$ are their fluctuations.

To coarse grain the fluctuating membrane, we rewrite the free energy as a functional of the average $(\bar{\mathbf{P}}, \bar{h})$:

$$\text{Free energy} = -(\beta)^{-1} \log Z = F[\bar{\mathbf{P}}, \bar{h}], \tag{9}$$

where $\beta = 1/k_B T$, and the partition function and the average $(\bar{\mathbf{P}}, \bar{h})$ are defined as

$$Z = \int [Dh, D\mathbf{P}] e^{-\beta H[\mathbf{P}, h]} \quad \text{and} \quad (\bar{\mathbf{P}}, \bar{h}) = \frac{1}{Z} \int [Dh, D\mathbf{P}](\mathbf{P}, h) e^{-\beta H[\mathbf{P}, h]}. \tag{10}$$

Further, we assume that the free energy depends *locally* on $(\bar{\mathbf{P}}, \bar{h})$ and we are only interested in terms up to quadratics of $(\bar{\mathbf{P}}, \bar{h})$:

$$F[\bar{\mathbf{P}}, \bar{h}] \approx L^2 \tau^{\text{eff}} + \int_{\mathbb{S}} \Big[\frac{a^{\text{eff}}}{2}(\bar{P}_x^2 + \bar{P}_y^2) + \frac{a_z^{\text{eff}}}{2}\bar{P}_z^2 + f^{\text{eff}}\bar{P}_z \Delta\bar{h} + \frac{\kappa_b^{\text{eff}}}{2}(\Delta\bar{h})^2$$
$$+ \kappa_g^{\text{eff}} \det(\nabla\nabla\bar{h}) + \frac{\lambda^{\text{eff}}}{2}|\nabla\bar{h}|^2 \Big], \tag{11}$$

where the linear terms of $(\bar{\mathbf{P}}, \bar{h})$ are absent since $(\bar{\mathbf{P}}, \bar{h}) = 0$ is necessarily a stationary point of $F[\bar{\mathbf{P}}, \bar{h}]$. The free energy may in general depend on $(\bar{\mathbf{P}}, \bar{h})$ nonlocally and in a more complicated manner. We assume that the averages $(\bar{\mathbf{P}}, \bar{h})$ are "close" to (0,0) such that (9) may be safely used [12, 13]. The critical coarse-graining principle is that the above free energy dictates the macroscopic behavior of the flexoelectric membrane at a finite constant temperature as the Hamiltonian (3) does at the zero temperature if $|(\bar{\mathbf{P}}, \bar{h})| \sim \varepsilon << 1$; the effective materials constants τ^{eff}, a^{eff}, a_z^{eff}, $\kappa_b^{\text{eff}}, \kappa_g^{\text{eff}}, \lambda^{\text{eff}}$ are all we need to predict the macroscopic behavior of the flexoelectric membrane in external environment.

For brevity we introduce notation:

$$\mathbf{v} = \begin{bmatrix} P_x \\ P_y \\ P_z \\ h \end{bmatrix}, \quad \bar{\mathbf{v}} = \begin{bmatrix} \bar{P}_x \\ \bar{P}_y \\ \bar{P}_z \\ \bar{h} \end{bmatrix}, \quad \mathbf{v}' = \begin{bmatrix} P'_x \\ P'_y \\ P'_z \\ h' \end{bmatrix}.$$

Then the Hamiltonian can be rewritten as

$$H[\mathbf{v}] = H_0[\mathbf{v}'] + \triangle H[\mathbf{v}', \bar{\mathbf{v}}], \qquad \triangle H[\mathbf{v}', \bar{\mathbf{v}}] = H[\mathbf{v}] - H_0[\mathbf{v}'] = L[\mathbf{v}', \bar{\mathbf{v}}] + \frac{1}{2}Q[\mathbf{v}', \bar{\mathbf{v}}] + \cdots, \tag{12}$$

where $H_0[\mathbf{v}'] = H[\mathbf{v}' + \eta\bar{\mathbf{v}}]\big|_{\eta=0}$ is independent of $\bar{\mathbf{v}}$, $L[\mathbf{v}', \bar{\mathbf{v}}]$ is a linear functional of $\bar{\mathbf{v}}$

$$L[\mathbf{v}', \bar{\mathbf{v}}] = \frac{dH[\mathbf{v}' + \eta\bar{\mathbf{v}}]}{d\eta}\bigg|_{\eta=0}, \tag{13}$$

4

and $Q[\mathbf{v}', \bar{\mathbf{v}}]$ is a quadratic functional of $\bar{\mathbf{v}}$

$$Q[\mathbf{v}', \bar{\mathbf{v}}] = \frac{d^2 H[\mathbf{v}' + \eta\bar{\mathbf{v}}]}{d\eta^2}\Bigg|_{\eta=0}. \tag{14}$$

In Appendix 2, we calculate explicitly $L[\mathbf{v}', \bar{\mathbf{v}}]$ and $Q[\mathbf{v}', \bar{\mathbf{v}}]$ up to the quadratic terms of $\mathbf{v}'$. Let

$$Z_0 = \int [D\mathbf{v}']e^{-\beta H_0[\mathbf{v}']}.$$

Assuming that $\beta\Delta H[\mathbf{v}', \bar{\mathbf{v}}] \sim \eta$ with $\eta << 1$, we find that the partition function (10) can be well approximated as

$$Z = \int [D\mathbf{v}']e^{-\beta H_0[\mathbf{v}']}(1 - \beta\Delta H[\mathbf{v}', \bar{\mathbf{v}}]) + o(\eta),$$

and hence the free energy is approximately given by

$$F[\bar{\mathbf{v}}] = -(\beta)^{-1}\log Z = -(\beta)^{-1}\log\left\{ Z_0 - \beta\int [D\mathbf{v}']e^{-\beta H_0[\mathbf{v}']}\Delta H[\mathbf{v}', \bar{\mathbf{v}}] + o(\eta)\right\}$$
$$= -(\beta)^{-1}\log Z_0 + \left\langle\Delta H[\mathbf{v}', \bar{\mathbf{v}}]\right\rangle + o(\eta), \tag{15}$$

where

$$\langle \cdot \rangle = \frac{1}{Z_0}\int [D\mathbf{v}']e^{-\beta H_0[\mathbf{v}']}(\cdot) \tag{16}$$

denotes the ensemble average. By (15) we conclude that the renormalized flexoelectric membrane is completely determined by the *fluctuations* of the system.

4   Statistical Mechanics of the Thermal Fluctuations

The ensemble averages (16) are intractable for a general nonlinear Hamiltonian. Two simplifications will be employed below: (1) the nonlinear Hamiltonian $H_0[\mathbf{v}']$ is expanded and truncated with terms up to quadratics of $\mathbf{v}'$, and (2) the nonlinear integration measure $[D\mathbf{v}']$ is replaced by the "naive measure" [30]:

$$[D\mathbf{v}']_{\text{naive}} = \prod_{m,i} \frac{1}{\lambda_m} d(\mathbf{v}')_{mi}, \tag{17}$$

where the subscript $m = 1, \cdots, 4$ denotes the mth component of $\mathbf{v}$, i labels the grid points on $\mathbb{S}$, and λ_m is some normalization constants such that $(\mathbf{v}')_m/\lambda_m$, is dimensionless. Then the quantity of interest $\langle\Delta H[\mathbf{v}', \bar{\mathbf{v}}]\rangle$ becomes a standard Gaussian integral and may be successively evaluated by the equipartition theorem and Isserlis' (or Wick's) theorem. To proceed, we recall that upon neglecting higher-order terms than δ^2 ($|\mathbf{v}'| \sim \delta$), the Hamiltonian $H_0[\mathbf{v}']$ can be written as

$$H_0[\mathbf{v}'] = \lambda L^2 + \int_{\mathbb{S}}\left[\frac{1}{2}a[(P'_x)^2 + (P'_x)^2] + \frac{1}{2}a_z(P'_z)^2 + fP'_z\Delta h' + \frac{1}{2}\kappa_b(\Delta h')^2\right.$$
$$\left. +\kappa_g\det(\nabla\nabla h') + \frac{1}{2}\lambda|\nabla h'|^2\right]. \tag{18}$$

By Fourier transformations we have

$$\mathbf{v}' = \sum_{\mathbf{q} \in \mathbb{K}} \hat{\mathbf{v}}'_{\mathbf{q}} e^{i\mathbf{q}\cdot\mathbf{r}}, \qquad \hat{\mathbf{v}}'_{\mathbf{q}} = \frac{1}{L^2} \int_{\mathbb{S}} \mathbf{v}' e^{-i\mathbf{q}\cdot\mathbf{r}} d\mathbf{r},$$

where, as usual, to avoid divergence, we introduce physically motivated cut-offs for the wave vectors which then satisfy $q := |\mathbf{q}| \in [q_{\min}, q_{\max}]$, i.e.,

$$\mathbb{K} = \{\mathbf{q} : \mathbf{q} = \frac{2\pi}{L}(n_x, n_y), \ n_x, n_y \in \mathbb{Z}, \ |\mathbf{q}| \in [q_{\min}, q_{\max}]\}.$$

The cut-off maximum wave vector $q_{\max}$ is related to the membrane thickness and we may set, $q_{\max} = \frac{2\pi}{t}$ while $q_{\min}$ is dictated by the macroscopic coarse-graining length scale, which we identify with L, i.e. $q_{\min} = \frac{2\pi}{L}$. By Parseval's theorem we can rewrite the Hamilton (18) in $\mathbf{q}$-space as

$$\begin{aligned}
H_0[\mathbf{v}'] = \lambda L^2 + L^2 \sum_{\mathbf{q} \in \mathbb{K}} &\left\{ \frac{a}{2}[|(\hat{\mathbf{P}}'_{\mathbf{q}})_x|^2 + |(\hat{\mathbf{P}}'_{\mathbf{q}})_y|^2] + \frac{a_z}{2}|(\hat{\mathbf{P}}'_{\mathbf{q}})_z|^2 - fq^2(\hat{\mathbf{P}}'_{\mathbf{q}})_z \hat{h}'_{-\mathbf{q}} \right. \\
&\left. + \frac{1}{2}\kappa_b q^4 |\hat{h}'_{\mathbf{q}}|^2 + \frac{1}{2}\lambda q^2 |\hat{h}'_{\mathbf{q}}|^2 \right\} \\
=: \lambda L^2 + L^2 &\sum_{\mathbf{q} \in \mathbb{K}} \frac{1}{2} \hat{\mathbf{v}}^*_{\mathbf{q}} \cdot \mathbf{M}_q \hat{\mathbf{v}}_{\mathbf{q}},
\end{aligned} \tag{19}$$

where the superscript $*$ denotes the complex conjugate,

$$\mathbf{M}_q = \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & a_z & -fq^2 \\ 0 & 0 & -fq^2 & \lambda q^2 + \kappa_b q^4 \end{bmatrix}.$$

We remark that, as expected, the term $\kappa_g \det(\nabla\nabla h')$ does not contribute to the right-hand side of (19) due to the Gauss-Bonnet theorem. At a finite temperature T, by the equipartition theorem we have that for $i, j = 1, \cdots, 4$ and $\mathbf{q}, \mathbf{q}' \in \mathbb{K}$,

$$\left\langle (\hat{\mathbf{v}}_{\mathbf{q}'})_i (\hat{\mathbf{v}}^*_{\mathbf{q}})_j \right\rangle = \frac{1}{L^2 \beta} \delta_{\mathbf{q}'\mathbf{q}} (\mathbf{M}_q^{-1})_{ij}. \tag{20}$$

We now evaluate the key quantities on the right hand side of (15):

$$\left\langle \Delta H[\mathbf{v}', \bar{\mathbf{v}}] \right\rangle = \left\langle L[\mathbf{v}', \bar{\mathbf{v}}] + \frac{1}{2} Q[\mathbf{v}', \bar{\mathbf{v}}] + \cdots \right\rangle.$$

By (9) we see that it is sufficient to compute $\left\langle L[\mathbf{v}', \bar{\mathbf{v}}] \right\rangle$ and $\left\langle Q[\mathbf{v}', \bar{\mathbf{v}}] \right\rangle$ to extract the effective materials constants $a^{\mathrm{eff}}, a_z^{\mathrm{eff}}, \kappa_b^{\mathrm{eff}}, \kappa_g^{\mathrm{eff}}, \lambda^{\mathrm{eff}}$ in (9). Comparing (15) with (9) immediately yields that

$$\left\langle L[\mathbf{v}', \bar{\mathbf{v}}] \right\rangle = 0,$$

which can also be confirmed by subsequent explicit evaluations. By (14), (31), (32), (33) and

identities summarized in Appendix 2 and Appendix 3, we obtain that

$$
\begin{aligned}
\langle \Delta H[\mathbf{v}', \bar{\mathbf{v}}] \rangle = \frac{1}{2} \langle Q[\mathbf{v}', \bar{\mathbf{v}}] \rangle \approx \frac{1}{2} \int_{\mathbb{S}} \Big\{ & a(\bar{P}_x^2 + \bar{P}_y^2) + a_z \bar{P}_z^2 + \frac{1}{2} |\nabla \bar{h}|^2 \langle a[(P_x')^2 + (P_y')^2] + a_z (P_z')^2 \rangle \\
& + \frac{1}{2} \langle |\nabla h'|^2 \rangle [a(\bar{P}_x^2 + \bar{P}_y^2) + a_z \bar{P}_z^2] \\
& + f \Big[\bar{P}_z \Delta \bar{h} - \bar{P}_z \Delta \bar{h} \langle |\nabla h'|^2 \rangle - \langle P_z' \Delta h' \rangle |\nabla \bar{h}|^2 \Big] \\
& + \kappa_b \Big[(\Delta \bar{h})^2 - \frac{3}{2} (\Delta \bar{h})^2 \langle |\nabla h'|^2 \rangle - \frac{3}{2} \langle (\Delta h')^2 \rangle |\nabla \bar{h}|^2 \Big] \\
& + \kappa_g \Big[\det(\nabla \nabla \bar{h}) - \frac{3}{2} \langle |\nabla h'|^2 \rangle \det(\nabla \nabla \bar{h}) \Big] \\
& + \lambda \Big[\frac{1}{2} |\nabla \bar{h}|^2 - \frac{3}{8} |\nabla \bar{h}|^2 \langle |\nabla h'|^2 \rangle \Big] \Big\}.
\end{aligned}
$$

Comparing (15) with (9), by the above equation and Appendix 3, we finally find that:

$$
\begin{aligned}
a^{\text{eff}} &= a(1 + \frac{1}{2} \langle |\nabla h'|^2 \rangle) = a\Big[1 + \frac{\eta \alpha}{4\pi}\Big] \approx a\Big[1 + \frac{k_B T}{4\pi(\kappa_b - f^2/a_z)} \ln \frac{L}{t}\Big], \\
a_z^{\text{eff}} &= a_z(1 + \frac{1}{2} \langle |\nabla h'|^2 \rangle) = a_z\Big[1 + \frac{\eta \alpha}{4\pi}\Big] \approx a_z\Big[1 + \frac{k_B T}{4\pi(\kappa_b - f^2/a_z)} \ln \frac{L}{t}\Big], \\
f^{\text{eff}} &= f(1 - \langle |\nabla h'|^2 \rangle) = f\Big[1 - \frac{\eta \alpha}{2\pi}\Big] \approx f[1 - \frac{k_B T}{2\pi(\kappa_b - f^2/a_z)} \ln \frac{L}{t}], \\
\kappa_b^{\text{eff}} &= \kappa_b\Big[1 - \frac{3}{2} \langle |\nabla h'|^2 \rangle\Big] = \kappa_b\Big[1 - \frac{3\eta \alpha}{4\pi}\Big] \approx \kappa_b[1 - \frac{3k_B T}{4\pi(\kappa_b - f^2/a_z)} \ln \frac{L}{t}], \\
\kappa_g^{\text{eff}} &= \kappa_g\Big[1 - \frac{3}{2} \langle |\nabla h'|^2 \rangle\Big] = \kappa_g\Big[1 - \frac{3\eta \alpha}{4\pi}\Big] \approx \kappa_g[1 - \frac{3k_B T}{4\pi(\kappa_b - f^2/a_z)} \ln \frac{L}{t}], \\
\lambda^{\text{eff}} &= \lambda\Big[1 - \frac{3}{8} \langle |\nabla h'|^2 \rangle\Big] + \langle \frac{a}{2}[(P_x')^2 + (P_y')^2] + \frac{a_z}{2}(P_z')^2 \rangle - f \langle P_z' \Delta h' \rangle - \frac{3}{2} \kappa_b \langle |\nabla h'|^2 \rangle \\
&= \lambda\Big[1 - \frac{3\eta \alpha}{16\pi}\Big] + \frac{k_B T}{4\pi} \Big[\int_{q_{\min}}^{q_{\max}} \frac{(\nu^2 + \frac{q^2}{1-\theta})q \, dq}{\nu^2 + q^2} + q_{\max}^2 - q_{\min}^2 \Big] + \kappa_b \eta \xi^2 \frac{2\theta - 3}{4\pi}.
\end{aligned}
$$

5 Discussion

A few words regarding our derivations are in order. The use of equipartition theorem implies that the calculated correlations (34) are accurate to the order of $k_B T$. To improve the order of accuracy, two considerations are necessary: (1) the expansion of $Q[\mathbf{v}, \mathbf{v}']$ in terms of $\mathbf{v}'$ should be up to the quartic terms of $\mathbf{v}'$, and (2) the naive measure (17) shall be corrected to account for the "Faddeev-Popov" factor [32] and the "Liouville" factor [33]. The necessity and motivation of such corrections are articulated in Cai et. al. [30] in the absence of flexoelectric effects. Moreover, an illustrative calculation concerning renormalization of a non-flexoelectric membrane is presented in [14] by directly evaluating the ensemble average of Hamiltonian $\langle H \rangle$. By definition the free energy F is given by

$$
F = U - TS, \quad U = \langle H \rangle, \quad S(T) = S_0 + S_1 T + \cdots,
$$

where U is the internal energy, S is the entropy and S_0 is some $(\bar{\mathbf{P}}, \bar{h})$-independent constant. From the above equation, we see that, indeed, for effective material constants up to the order of $k_B T$ it

is sufficient to evaluate $\langle H \rangle$. The danger, however, lies in the zeroth-order term. From the above equation we see that F and U differ by $-TS_0$ for $(\bar{\mathbf{P}}, \bar{h}) = 0$; the renormalized surface "tension" τ^{eff} defined by (11) cannot be calculated by simply evaluating the ensemble average of Hamiltonian.

Our results are predicated on the the Monge representation and accordingly we have assumed that the fluctuations are over a state that is, on average, flat. This precludes us from accounting membrane states that reflect overhands which can be important in certain biophysical phenomena. A fully nonlinear reparametrization invariant extension of our work is likely to be an interesting future research direction.

Our results have ramifications for the interpretation of measured experimental values of the flexoelectric coefficient as well as the bending rigidity. As example, we may write the renormalized bending modulus in the following way, $\kappa_b^{\text{eff}} \approx \kappa_b - \phi \cdot \frac{3k_B T}{4\pi} \ln \frac{L}{t}$. Here $\phi = 1$ in absence of flexoelectricity and $\phi = \frac{\kappa_b}{\kappa_b - f^2/a_z} > 1$ when it is present. The departure from unity depends on the factor f^2/a_z. Taking a value of the flexoelectricity coefficient, $f_e = 0.31 \times 10^{-19} C$ (which is on the lower side found experimentally, [20]), we find that $f^2/a_z = f_e^2 a_z \approx 10 k_B T$. With κ_b lying between $10 - 20 k_B T$, this will lead to significant softening. In the numerical calculations, we have chosen a bilayer thickness of 5 nm and room temperature conditions ($25^o C$). We note here an interesting peculiarity. Stability analysis (5) requires that $\kappa_b > f^2/a_z$. Experimentally determined flexoelectric coefficients [20] lie between $f_e \approx 0.3 \times 150 \times 10^{-19} C$ depending on the degree of partial charges on the lipids and the ionic conditions. For some of the larger values of the flexoelectric coefficient, superficially at least, there appears to be an apparent violation of the stability condition (assuming that the bare bending modulus κ_b is strictly between $10 - 20 k_B T$). A possibility is that the bare modulus itself is different under those conditions—experiments only measure κ_b^{eff}. This paradox may be therefore, *possibly* reconciled by noting that, as explained quite well by [20], presence of mechanisms such as surface charges may dramatically change the apparent flexoelectricity of the membranes and under such cases, perhaps a revised theoretical framework is needed (which will presumably have different stability conditions). Stated differently, the higher values of flexoelectricity coefficients reported in the literature may not be intrinsic properties but rather some renormalized values. As case in point, a recent numerical study predicated on non-linear electrostatics [34], indicated that the flexoelectric coefficient of a prototypical lipid bilayter is, $f_e \approx 0.095e$ and this leads to f^2/a_z to be roughly $2.8 k_B T$. We should point out here that in works related to the outer hair cell e.g. Harland et. al. [34], the membrane stiffness is quite high and thermal fluctuations aspect of our work is not likely to play a major role there. In summary, we conclude that flexoelectricity is likely to be a significant contributor to the mechanical and dielectric properties of lipid bilayers and temperature appears to have a strong softening effect on flexoelectricity itself.

Appendix 1: Contribution of the nonlocal electrostatic energy

By Toupin (1956) [36], the free energy of a *bulk* rigid polarizable body Ω consists of

$$H[\tilde{\mathbf{P}}] = \int_\Omega W(\tilde{\mathbf{P}}) + H_{\text{field}}[\mathbf{P}], \qquad H_{\text{field}}[\mathbf{P}] = \frac{\epsilon_0}{2} \int_{\mathbb{R}^3} |\mathbf{E}|^2, \tag{21}$$

where $\tilde{\mathbf{P}}$ is the polarization per unit volume, the first term reflects that energy is needed to polarize in the body, and the second term is the energy associated with the electric field $\mathbf{E}$ induced by the polarization. If the medium of the body is a linear dielectric medium of permittivity ϵ, the internal

energy density function $W(\mathbf{P})$ is given by [36]:

$$W(\tilde{\mathbf{P}}) = \frac{1}{2}\tilde{a}|\tilde{\mathbf{P}}|^2, \qquad \tilde{a} = \frac{1}{\epsilon - \epsilon_0}. \tag{22}$$

We now consider a thin membrane $\mathbb{S} = (0, L)^2$ of thickness $t << L$. The polarization on the membrane inevitably induces an electric field penetrating into the entire space. Let $\mathbf{P}(x, y)$ denote the polarization per unit area (and hence $\tilde{\mathbf{P}} = \frac{1}{t}\mathbf{P}$) on the membrane $\mathbb{S}$ and

$$\chi_t(z) = 1 \qquad \text{if } z \in [-\frac{t}{2}, \frac{t}{2}], \qquad = 0 \text{ otherwise.}$$

Since the lateral dimension of the membrane is much larger than the thickness ($L >> t$) and the polarization $\mathbf{P}$ is assumed to be independent of z, the electric field near to the membrane is well approximated by that determined by the following Maxwell's equation:

$$\text{div}(-\epsilon_0 \nabla \xi(x, y, z) + \frac{1}{t}\mathbf{P}(x, y)\chi_t(z)) = 0 \qquad \text{in } \mathbb{R}^3. \tag{23}$$

In the above equation, we have ignore the fact that the membrane is deformed. Then the associated field energy induced by the polarization is given by

$$H_{\text{field}}[\mathbf{P}] = \frac{\epsilon_0}{2}\int_{\mathbb{S}}\int_{-\infty}^{\infty}|\nabla\xi|^2 dz d\mathbf{x}. \tag{24}$$

We now solve (23) by Fourier method. For brevity denote by $\mathbf{x} = (x, y)$ and $\mathbf{k} = (k_x, k_y)$. Let

$$\hat{\mathbf{P}}_{\mathbf{k}} = \frac{1}{L^2}\int_{\mathbb{S}}\mathbf{P}(x, y)e^{-i\mathbf{k}\cdot\mathbf{x}}d\mathbf{x}, \qquad \hat{\xi}_{\mathbf{k}}(z) = \frac{1}{L^2}\int_{\mathbb{S}}\xi(\mathbf{x}, z)e^{-i\mathbf{k}\cdot\mathbf{x}}d\mathbf{x} \tag{25}$$

be the Fourier coefficients. Then the inversion theorem implies that

$$\mathbf{P}(\mathbf{x}) = \sum_{\mathbf{k}\in\mathbb{K}}\hat{\mathbf{P}}_{\mathbf{k}}e^{i\mathbf{k}\cdot\mathbf{x}}, \qquad \xi(\mathbf{x}, z) = \sum_{\mathbf{k}\in\mathbb{K}}\hat{\xi}_{\mathbf{k}}(z)e^{i\mathbf{k}\cdot\mathbf{x}}. \tag{26}$$

Inserting (26) into (23) we find that

$$\epsilon_0|\mathbf{k}|^2\hat{\xi}_{\mathbf{k}}(z) - \epsilon_0\frac{d^2}{dz^2}\hat{\xi}_{\mathbf{k}}(z) + i\frac{1}{t}\hat{\mathbf{P}}_{\mathbf{k}}\cdot\mathbf{k}\chi_t(z) + \frac{1}{t}(\hat{\mathbf{P}}_{\mathbf{k}})_z\frac{d}{dz}\chi_t(z) = 0 \qquad \forall z \in \mathbb{R}, \ \mathbf{k} \in \mathbb{K}.$$

The above linear ordinary differential equation has two source terms; the solution can be conveniently expressed as

$$\hat{\xi}_{\mathbf{k}}(z) = \frac{(\hat{\mathbf{P}}_{\mathbf{k}})_z}{t\epsilon_0|\mathbf{k}|^2}\int_{\mathbb{R}}G_{\mathbf{k}}(z - z')\frac{d}{dz'}\chi_t(z')dz' + \frac{i\hat{\mathbf{P}}_{\mathbf{k}}\cdot\mathbf{k}}{t\epsilon_0|\mathbf{k}|^2}\int_{\mathbb{R}}G_{\mathbf{k}}(z - z')\chi_t(z')dz', \tag{27}$$

where

$$G_{\mathbf{k}}(z) = \begin{cases} \frac{1}{2}|\mathbf{k}|e^{-|\mathbf{k}|z} & \text{if } z \geq 0, \\ \frac{1}{2}|\mathbf{k}|e^{|\mathbf{k}|z} & \text{if } z < 0. \end{cases}$$

is the Green's function satisfying

$$-\frac{1}{|\mathbf{k}|^2}\frac{d^2}{dz^2}G_{\mathbf{k}}(z) + G_{\mathbf{k}}(z) + \delta(0) = 0 \qquad \forall z \in \mathbb{R}.$$

9

Direct integration of (27) yields

$$\hat{\xi}_{\mathbf{k}}(z) = \begin{cases} \frac{(\hat{\mathbf{P}}_{\mathbf{k}})_z}{2t\epsilon_0|\mathbf{k}|}(e^{|\mathbf{k}|t/2} - e^{-|\mathbf{k}|t/2})e^{-|\mathbf{k}|z} - \frac{i\hat{\mathbf{P}}_{\mathbf{k}}\cdot\mathbf{k}}{2t\epsilon_0|\mathbf{k}|^2}(e^{|\mathbf{k}|t/2} + e^{-|\mathbf{k}|t/2})e^{-|\mathbf{k}|z} & \text{if } z \geq t/2, \\ \frac{(\hat{\mathbf{P}}_{\mathbf{k}})_z}{2t\epsilon_0|\mathbf{k}|}(e^{|\mathbf{k}|z} - e^{-|\mathbf{k}|z})e^{-|\mathbf{k}|t/2} - \frac{i\hat{\mathbf{P}}_{\mathbf{k}}\cdot\mathbf{k}}{t\epsilon_0|\mathbf{k}|^2}[1 - \frac{e^{-|\mathbf{k}|t/2}}{2}(e^{|\mathbf{k}|z} + e^{-|\mathbf{k}|z})] & \text{if } -t/2 \leq z \leq t/2, \\ -\frac{(\hat{\mathbf{P}}_{\mathbf{k}})_z}{2t\epsilon_0|\mathbf{k}|}(e^{|\mathbf{k}|t/2} - e^{-|\mathbf{k}|t/2})e^{|\mathbf{k}|z} - \frac{i\hat{\mathbf{P}}_{\mathbf{k}}\cdot\mathbf{k}}{2t\epsilon_0|\mathbf{k}|^2}(e^{|\mathbf{k}|t/2} + e^{-|\mathbf{k}|t/2})e^{|\mathbf{k}|z} & \text{if } z \leq -t/2. \end{cases}$$

$$(28)$$

Therefore, the electric field energy defined by (24) is given by

$$H_{\text{field}}[\mathbf{P}] = \frac{\epsilon_0}{2}\int_{\mathbb{S}}\int_{-\frac{t}{2}}^{\frac{t}{2}}\frac{1}{t}\mathbf{P}\cdot\nabla\xi dz d\mathbf{x} = \sum_{\mathbf{k}\in\mathbb{K}}\frac{L^2}{2t}\int_{-\frac{t}{2}}^{\frac{t}{2}}\left\{(\hat{\mathbf{P}}_{\mathbf{k}})_z^*\frac{d\hat{\xi}_{\mathbf{k}}}{dz} + i\hat{\mathbf{P}}_{\mathbf{k}}^*\cdot\mathbf{k}\hat{\xi}_{\mathbf{k}}\right\}dz$$

$$= \sum_{\mathbf{k}\in\mathbb{K}}\frac{L^2}{2t\epsilon_0}\left\{|(\hat{\mathbf{P}}_{\mathbf{k}})_z|^2\frac{1 - e^{-t|\mathbf{k}|}}{t|\mathbf{k}|} + \frac{|\hat{\mathbf{P}}_{\mathbf{k}}\cdot\mathbf{k}|^2}{|\mathbf{k}|^2}[1 - \frac{1}{t|\mathbf{k}|}(1 - e^{-t|\mathbf{k}|})]\right\}. \quad (29)$$

By (29), we observe that if $t|\mathbf{k}| << 1$, i.e., the typical wavelengths of the fluctuation are much larger than the thickness of the membrane, the field energy associated with the in-plane polarization, i.e., $\sum_{\mathbf{k}\in\mathbb{K}}\frac{|\hat{\mathbf{P}}_{\mathbf{k}}\cdot\mathbf{k}|^2}{|\mathbf{k}|^2}[1 - \frac{1}{t|\mathbf{k}|}(1 - e^{-t|\mathbf{k}|})]$, is of order $(t|\mathbf{k}|)^2$ whereas the contribution from the out-of-plane polarization, to the leading order, is given by

$$H_{\text{field}}[\mathbf{P}] = \sum_{\mathbf{k}\in\mathbb{K}}\frac{L^2}{2t\epsilon_0}|(\hat{\mathbf{P}}_{\mathbf{k}})_z|^2 + O(t|\mathbf{k}|).$$

Keeping only the leading order term, we find that the nonlocal field energy can be rewritten as

$$H_{\text{field}}[\mathbf{P}] \approx \sum_{\mathbf{k}\in\mathbb{K}}\frac{L^2}{2t\epsilon_0}|(\hat{\mathbf{P}}_{\mathbf{k}})_z|^2 = \frac{1}{2t\epsilon_0}\int_{\mathbb{S}}|P_z|^2. \quad (30)$$

Inserting the above equation into (21), by (22) we find that for the membrane the contribution to the Hamiltonian by polarization *alone* is given by

$$\int_{\mathbb{S}}[a(P_x^2 + P_y^2) + a_z P_z^2],$$

where a, a_z are given by (4).

Appendix 2: Calculation of $L[\mathbf{v}', \mathbf{v}]$ and $Q[\mathbf{v}', \mathbf{v}]$

To evaluate the free energy by (15), it is necessary to compute $\Delta H[\mathbf{v}', \mathbf{v}] = H[\mathbf{v}] - H_0[\mathbf{v}']$ defined by (12). To proceed, we expand and truncate the Hamiltonian in terms of $(\bar{\mathbf{P}}, \bar{h})$ and $(\mathbf{P}', h')$. For convenience, assume that $|h'| \sim |\mathbf{P}'| \sim \varepsilon$, $|\bar{h}| \sim |\bar{\mathbf{P}}| \sim \delta$, and $\varepsilon, \delta << 1$. Neglecting higher-order terms than ε^2 or δ^2, by straightforward yet tedious calculations we find that

$$J_h = 1 + \frac{1}{2}|\nabla\bar{h}|^2 + \frac{1}{2}|\nabla h'|^2 + \nabla\bar{h}\cdot\nabla h' - \frac{3}{4}(\nabla\bar{h}\cdot\nabla h')^2 + o(\varepsilon^2, \delta^2),$$

$$J_h^{-3} = 1 - \frac{3}{2}|\nabla\bar{h}|^2 - \frac{3}{2}|\nabla h'|^2 - 3\nabla\bar{h}\cdot\nabla h' + \frac{45}{4}(\nabla\bar{h}\cdot\nabla h')^2 + o(\varepsilon^2, \delta^2),$$

$$K_h = \Delta(\bar{h} + h') - \frac{1}{2}(\Delta\bar{h}|\nabla h'|^2 + \Delta h'|\nabla\bar{h}|^2) - \Delta(\bar{h} + h')\nabla\bar{h}\cdot\nabla h' \quad (31)$$

$$\quad - \nabla h'\cdot(\nabla\nabla\bar{h})\nabla h' - \nabla\bar{h}\cdot(\nabla\nabla h')\nabla\bar{h} - 2\nabla h'\cdot[\nabla\nabla(\bar{h} + h')]\nabla\bar{h} + o(\varepsilon^2, \delta^2),$$

$$\det(\nabla\nabla h) = \det(\nabla\nabla\bar{h}) + \det(\nabla\nabla h') + \Delta\bar{h}\Delta h' - \nabla\nabla\bar{h}\cdot\nabla\nabla h'.$$

Therefore,

$$
\begin{aligned}
J_h(K_h)^2 = &[\Delta(\bar{h}+h')]^2 - 2\Delta\bar{h}[\frac{1}{2}\Delta\bar{h}|\nabla h'|^2 + \nabla h' \cdot (\nabla\nabla\bar{h})\nabla h'] \\
&-2\Delta h'[\frac{1}{2}\Delta h'|\nabla\bar{h}|^2 + \nabla\bar{h} \cdot (\nabla\nabla h')\nabla\bar{h}] \\
&+\frac{1}{2}|\nabla h'|^2(\Delta\bar{h})^2 + \frac{1}{2}|\nabla\bar{h}|^2(\Delta h')^2 - 2\Delta\bar{h}\Delta h'\nabla h' \cdot \nabla\bar{h} \\
&-4\Delta\bar{h}\nabla h' \cdot (\nabla\nabla h')\nabla\bar{h} - 4\Delta h'\nabla h' \cdot (\nabla\nabla\bar{h})\nabla\bar{h} + o(\varepsilon^2,\delta^2), \quad (32)
\end{aligned}
$$

$$
\begin{aligned}
J_h^{-3}G_h = &\det(\nabla\nabla\bar{h}) + \det(\nabla\nabla h') + \Delta\bar{h}\Delta h' - \nabla\nabla\bar{h} \cdot \nabla\nabla h' - \frac{3}{2}|\nabla\bar{h}|^2\det(\nabla\nabla h') \\
&-\frac{3}{2}|\nabla h'|^2\det(\nabla\nabla\bar{h}) - 3(\nabla\bar{h} \cdot \nabla h')[\Delta\bar{h}\Delta h' - \nabla\nabla\bar{h} \cdot \nabla\nabla h'] + o(\varepsilon^2,\delta^2).
\end{aligned}
$$

Similarly, we have that

$$
|\mathbf{P}|^2 J_h = |\bar{\mathbf{P}}|^2 + 2\bar{\mathbf{P}} \cdot \mathbf{P'} + |\mathbf{P'}|^2 + \frac{1}{2}|\nabla\bar{h}|^2|\mathbf{P'}|^2 + \frac{1}{2}|\nabla h'|^2|\bar{\mathbf{P}}|^2 + 2(\nabla h' \cdot \nabla\bar{h})\bar{\mathbf{P}} \cdot \mathbf{P'} + o(\varepsilon^2,\delta^2),
$$

$$
\begin{aligned}
P_z K_h = &\bar{P}_z\Big[\Delta(\bar{h}+h') - \frac{1}{2}\Delta\bar{h}|\nabla h'|^2 - \nabla h' \cdot (\nabla\nabla\bar{h})\nabla h' \\
&-2\nabla h' \cdot (\nabla\nabla h')\nabla\bar{h} - \Delta h'\nabla\bar{h} \cdot \nabla h'\Big] + P'_z\Big[\Delta(\bar{h}+h') - \frac{1}{2}\Delta h'|\nabla\bar{h}|^2 - \nabla\bar{h} \cdot (\nabla\nabla h')\nabla\bar{h} \\
&-2\nabla h' \cdot (\nabla\nabla\bar{h})\nabla\bar{h} - \Delta\bar{h}\nabla\bar{h} \cdot \nabla h'\Big] + o(\varepsilon^2,\delta^2), \quad (33)
\end{aligned}
$$

$$
P_x h_{,x} K_h = \bar{P}_x\bar{h}_{,x}\Delta h' + (\bar{P}_x h'_{,x} + P'_x\bar{h}_{,x})\Delta(\bar{h}+h') + P'_x h'_{,x}\Delta\bar{h} + o(\varepsilon^2,\delta^2).
$$

Appendix 3: Evaluation of the ensemble averages

Define constants $\theta, \eta, \nu, \alpha, \xi$ as

$$
\theta = \frac{f^2}{a\kappa_b}, \qquad \eta = \frac{k_B T}{\kappa_b(1-\theta)}, \qquad \nu^2 = \frac{\lambda}{\kappa_b(1-\theta)},
$$

$$
\alpha = \int_{q_{min}}^{q_{max}} \frac{q\,dq}{\nu^2 + q^2} = \frac{1}{2}\log\frac{\nu^2 + q_{max}^2}{\nu^2 + q_{min}^2},
$$

$$
\xi^2 = \int_{q_{min}}^{q_{max}} \frac{q^3\,dq}{\nu^2 + q^2} = \frac{q_{max}^2 - q_{min}^2}{2} + \frac{\nu^2}{2}\log\frac{\nu^2 + q_{max}^2}{\nu^2 + q_{min}^2}.
$$

Note that ν, η, α are dimensionless while ν, ξ has the dimension of 1/length. By (20) the ensemble averages of fluctuations can be calculated term by term as follows:

$$
\begin{aligned}
\langle|\nabla h'|^2\rangle = \sum_{\mathbf{q}\in\mathbb{K}} q^2\langle|\hat{h}'_{\mathbf{q}}|^2\rangle &= \sum_{\mathbf{q}\in\mathbb{K}} \frac{k_B T a q^2}{L^2[\lambda a q^2 + (\kappa_b a - f^2)q^4]} \\
&\approx \frac{2\pi k_B T a}{L^2}\left(\frac{L}{2\pi}\right)^2\int_{q_{min}}^{q_{max}} \frac{q\,dq}{\lambda a + (\kappa_b a - f^2)q^2} = \frac{\eta\alpha}{2\pi}, \quad (34)
\end{aligned}
$$

$$
f\langle P'_z\Delta h'\rangle \approx \frac{-2\pi k_B T}{L^2}\left(\frac{L}{2\pi}\right)^2\int_{q_{min}}^{q_{max}} \frac{f^2 q^3\,dq}{\lambda a + (\kappa_b a - f^2)q^2} = \frac{-\kappa_b\eta\theta\xi^2}{2\pi},
$$

$$
\langle(\Delta h')^2\rangle \approx \frac{2\pi k_B T a}{L^2}\left(\frac{L}{2\pi}\right)^2\int_{q_{min}}^{q_{max}} \frac{q^3\,dq}{\lambda a + (\kappa_b a - f^2)q^2} = \frac{\eta\xi^2}{2\pi},
$$

11

$$\langle \mathbf{P}' \otimes \mathbf{P}' \rangle = \frac{k_B T}{2\pi} \operatorname{diag}\left[\frac{q_{\max}^2 - q_{\min}^2}{2a}, \ \frac{q_{\max}^2 - q_{\min}^2}{2a}, \int_{q_{\min}}^{q_{\max}} \frac{(\nu^2 + \frac{q^2}{1-\theta}) q \, dq}{a_z(\nu^2 + q^2)} \right].$$

By symmetry or (20) we immediately obtain

$$\langle \nabla h' \otimes \nabla h' \rangle = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \langle |\nabla h'|^2 \rangle, \qquad\qquad \langle P_z' \nabla\nabla h' \rangle = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \langle P_z' \Delta h' \rangle,$$

$$\langle \det(\nabla\nabla h') \rangle = 0, \qquad \langle (\nabla\nabla h')\nabla h' \rangle = 0, \qquad \langle \mathbf{P}' \otimes \nabla h' \rangle = 0. \tag{35}$$

The ensemble average of each term on the right-hand sides of (31)-(33), to the leading order, can be explicitly computed as follows. First, by (33) we have

$$\langle P_x^2 J_h \rangle \approx \bar{P}_x^2 + \langle (P_x')^2 \rangle + \frac{1}{2}|\nabla\bar{h}|^2 \langle (P_x')^2 \rangle + \frac{1}{2}\langle |\nabla h'|^2 \rangle \bar{P}_x^2. \tag{36}$$

Second, we have

$$\langle \mathbf{P} \cdot \mathbf{n}_h K_h J_h \rangle = \langle P_z K_h \rangle - \langle P_x h_{,x} K_h \rangle - \langle P_y h_{,y} K_h \rangle,$$

where, by (33), (34) and (35),

$$\langle P_z K_h \rangle \approx \bar{P}_z \Delta\bar{h} - \frac{1}{2}\bar{P}_z \Delta\bar{h}\langle |\nabla h'|^2 \rangle - \bar{P}_z (\nabla\nabla\bar{h}) \cdot \langle (\nabla h' \otimes \nabla h') \rangle$$

$$+ \langle P_z' \Delta h' \rangle - \frac{1}{2}\langle P_z' \Delta h' \rangle |\nabla\bar{h}|^2 - \nabla\bar{h} \cdot \langle P_z'(\nabla\nabla h') \rangle \nabla\bar{h}$$

$$= \bar{P}_z \Delta\bar{h} - \bar{P}_z \Delta\bar{h}\langle |\nabla h'|^2 \rangle + \langle P_z' \Delta h' \rangle - \langle P_z' \Delta h' \rangle |\nabla\bar{h}|^2, \tag{37}$$

$$\langle P_x h_{,x} K_h \rangle \approx 0, \qquad\qquad \langle P_y h_{,y} K_h \rangle \approx 0.$$

Finally, by (32), (34) and (35) we have

$$\langle (K_h)^2 J_h \rangle = (\Delta\bar{h})^2 + \langle (\Delta h')^2 \rangle - \frac{1}{2}(\Delta\bar{h})^2 \langle |\nabla h'|^2 \rangle - 2\Delta\bar{h}(\nabla\nabla\bar{h}) \cdot \langle \nabla h' \otimes \nabla h' \rangle$$

$$- \frac{1}{2}\langle (\Delta h')^2 \rangle |\nabla\bar{h}|^2 - 2\nabla\bar{h} \cdot \langle \Delta h' \nabla\nabla h' \rangle \nabla\bar{h}$$

$$= (\Delta\bar{h})^2 + \langle (\Delta h')^2 \rangle - \frac{3}{2}(\Delta\bar{h})^2 \langle |\nabla h'|^2 \rangle - \frac{3}{2}\langle (\Delta h')^2 \rangle |\nabla\bar{h}|^2, \tag{38}$$

$$\langle J_h^{-3} G_h \rangle = \det(\nabla\nabla\bar{h}) - \frac{3}{2}\langle |\nabla h'|^2 \rangle \det(\nabla\nabla\bar{h}),$$

$$\langle J_h \rangle = 1 + \frac{1}{2}|\nabla\bar{h}|^2 + \frac{1}{2}\langle |\nabla h'|^2 \rangle - \frac{3}{4}(\nabla\bar{h} \otimes \nabla\bar{h}) \cdot \langle \nabla h' \otimes \nabla h' \rangle.$$

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