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Journal of the Mechanics and Physics of Solids
54 (2006) 951–974
JOURNAL OF THE MECHANICS AND PHYSICS OF SOLIDS
www.elsevier.com/locate/jmps

Magnetostrictive composites in the dilute limit

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Received 9 March 2005; received in revised form 21 November 2005; accepted 22 November 2005

Abstract

We calculate the effective properties of a magnetostrictive composite in the dilute limit. The composite consists of well separated identical ellipsoidal particles of magnetostrictive material, surrounded by an elastic matrix. The free energy of the magnetostrictive particles is computed using the constrained theory of DeSimone and James [2002. A constrained theory of magnetoelasticity with applications to magnetic shape memory materials. J. Mech. Phys. Solids 50, 283–320], where application of an external field causes rearrangement of variants rather than rotation of the magnetization or elastic strain in a variant. The free energy of the composite has an elastic energy term associated with the deformation of the surrounding matrix and demagnetization terms. By using results from the constrained theory and from the Eshelby inclusion problem in linear elasticity, we show that the energy minimization problem for the composite can be cast as a quadratic programming problem. The solution of the quadratic programming problem yields the effective properties of Ni_2MnGa and Terfenol-D composite systems. Numerical results show that the average strain of the composite depends strongly on the particle shape, the applied stress, and the elastic modulus of the matrix.

Keywords: Constrained theory; Magnetostrictive composite; Eshelby problem

1. Introduction

Magnetostriction refers to the reversible deformation of a solid in response to an applied magnetic field. Magnetostriction occurs in most ferromagnetic materials. For example,

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0022-5096/$ - see front matter © 2005 Published by Elsevier Ltd.
doi:10.1016/j.jmps.2005.11.006

pure cobalt, nickel or iron have a maximum shape change of around 100 microstrain (Cullity, 1972), while the giant magnetostrictive alloy Terfenol-D has a magnetostriction of about 2000 microstrain (Clark, 1992). More recently, "Ferromagnetic Shape Memory" (FSM) materials such as Ni_2MnGa have been developed that can generate a magnetostrictive strain up to 50 times that of Terfenol-D. These materials work by coupling magnetic domains to martensite twin domains.

While Terfenol-D has been used in commercial applications, success of some of the newer materials has been limited by the fact that they are often quite brittle. Also, polycrystalline magnetostrictive materials usually have much lower magnetostrictive strains than their single crystal counterparts. Both of these difficulties can be addressed by developing magnetostrictive composites consisting of small, essentially single crystal particles embedded in a soft matrix, see for example McKnight and Carman (2001), McKnight (2002), and Pinkerton and Capehart (1997).

The main goal of this paper is to analyze the behavior of composites containing a dilute mixture of magnetostrictive particles. We describe the behavior of the particles using the constrained theory developed by DeSimone and James (2002). Constrained theory is derived from micromagnetics, see Landau and Lifshitz (1935) and Brown (1963), and is appropriate for magnetostrictive materials with high anisotropy and mobile variant interfaces. Specifically, we assume that because of these features of the magnetostrictive materials, the application of external fields causes only rearrangement of the variants rather than rotation of magnetization or elastic strain within a variant. This assumption greatly simplifies the energy minimization problem for an *isolated* ellipsoidal particle, leading to a finite dimensional quadratic programming problem. In this case, the constrained theory has been compared to experimental results, see e.g. Tickle et al. (1999), Cui (2002), and James and Wuttig (1998). These comparisons show that the constrained theory captures semi-quantitatively the main magnetostrictive behaviors of both FSM materials and giant magnetostrictive materials with high anisotropy and mobile variant interfaces.

The value of the constrained theory is that the final problem (a finite dimensional quadratic programming problem) is drastically easier than the original infinite dimensional variational problem. One can gain significant insight into the behavior of magnetostrictive materials by examining the structure and solutions of this problem. However, these results are restricted to ellipsoidal particles because the reduction to finite dimensions makes use of the fact that ellipsoidal regions have the property that a uniformly magnetized ellipsoid induces a uniform magnetic field inside the ellipsoid.

A second desirable feature of constrained theory is that it embodies what has been conceived as ideal ferromagnetic shape memory, that is, high magnetic anisotropy and mobile twin interfaces. (However, the constrained theory is equally valid for conventional magnetostrictive materials with high anisotropy and mobile domain walls.) Thus, it can be used to predict how the best materials of this type would behave, and it can set limitations on response. Currently, efforts are underway to significantly increase magnetic anisotropy in Fe_3Pd via ordering. If it would also retain its high mobility in the ordered state, then its behavior would be close to ideal. Even with the current materials, we believe our results provide significant insight into what stiffness of matrix, particle shape and applied stress can be used to enhance the field-induced strain in such a composite.

In this paper, we extend the results for an isolated magnetostrictive particle to the case of a composite system where magnetostrictive particles are embedded in an elastic matrix.

This is possible because the special place of ellipsoidal particles in micromagnetics extends to linear elasticity. That is, it is well-known that in linear elasticity, a uniform eigenstrain inside an ellipsoidal region in an infinite medium induces uniform stress inside the ellipsoid, see Eshelby (1957). By using this result (Eshelby's solution) and strategies similar to those used by DeSimone and James (2002), we simplify the minimization problem for the composite system to be a quadratic programming problem. This idea that Eshelby's results allow one to solve not only linear problems for composites, but also problems in which the matrix is linear and the inclusion is nonlinear, goes back to Hill (1965); in simple terms the idea is that, if the nonlinear material inside the inclusion is equilibrated (or energy minimizing) under homogeneous conditions of stress and strain then, even though the stress and strain may be nonlinearly related, a solution can still typically be found (say, by the implicit function theorem). In the present case the inclusion is indeed nonlinear, but there is also the presence of the nonlocal magnetostatic energy. However, once it is realized that this energy also enjoys special properties for ellipsoids—namely, a minimizer of the magnetostatic energy of an ellipsoidal inclusion over all magnetizations with given average magnetization is constant—then Hill's idea can be used.

There have been several recent papers describing the behavior of magnetoelastic composites. Kankanala and Triantafyllidis (2004) have developed a macroscopic continuum model for such composites based on both continuum balance laws and variational methods. Borcea and Bruno (2001) introduced composite microstructure by considering the interactions between two ferromagnetic particles to develop a second-order (in volume fraction) model for composite properties. These works consider how conventional magnetic forces between particles affect the composite, and do not consider the magnetostrictive strain of the particle arising from configurational forces. Armstrong (2000) considered such particles, though he considered particle straining to arise from the rotation of magnetization and the associated strain inside a variant, in direct contrast to our use of constrained theory. Our approach is also different from that of Herbst et al. (1997), and Nan and Weng (1999), who assume a linear magneto-elastic constitutive equations for the magnetostrictive particles and predict the effective magneto-elastic constitutive equations of the composite by solving a mixture problem. The present theory has no phenomenological parameters. The material parameters needed here are: the linear elastic modulus tensor of the matrix, the saturation magnetization of one variant of martensite, the transformation strain matrix and the variant structure of the magnetic phase.

In Section 2, we review the constrained theory for an isolated magnetostrictive particle. In Section 3 we outline how the constrained theory can be adapted for the composite system and how the minimization problems can be reduced to quadratic programming problems for ellipsoidal particles. In Section 4, we connect the average strain of the composite system with the strain of the magnetostrictive particles and clarify some technical problems concerning the external mechanical loading device. In Section 5, using the theory presented in previous sections, we numerically evaluate the magnetostrictive properties of dilute composites with Ni_2MnGa and Terfenol-D inclusions.

The notation we use is as follows: The convention of summation over double indices is used unless stated otherwise. We denote by $\mathbf{a}\cdot\mathbf{b}$, $\mathbf{a},\mathbf{b}\in\mathbb{R}^{n\times m}$ the usual inner product in the Euclidean space $\mathbb{R}^{n\times m}$, i.e., in indices form, $a_{ip}b_{ip}$. A linear mapping $\mathbf{R}:\mathbb{R}^{n\times m}\mapsto\mathbb{R}^{n\times m}$ can be identified with an element in $\mathbb{R}^{n\times m\times n\times m}$, i.e. $\mathbf{Ra}=R_{ipjq}a_{jq}\in\mathbb{R}^{n\times m}\ \ \forall\mathbf{a}\in\mathbb{R}^{n\times m}$. The mapping $\mathbf{R}^{-1}:\mathbb{R}^{n\times m}\mapsto\mathbb{R}^{n\times m}$ denotes the inverse mapping of $\mathbf{R}$ if the linear mapping $\mathbf{R}$ is

invertible. There are cases that an element $\mathbf{S} \in \mathbb{R}^{n\times m\times n\times m}$ is not an invertible mapping from $\mathbb{R}^{n\times m}$ to $\mathbb{R}^{n\times m}$, but is invertible when restricted to a subspace of $\mathbb{R}^{n\times m}$. By $\mathbf{S}^{-1}$ we mean the inverse mapping of $\mathbf{S}$ restricted to this subspace of $\mathbb{R}^{n\times m}$. By $\fint_\Omega d\mathbf{x}$ we mean $\frac{1}{|\Omega|}\int_\Omega d\mathbf{x}$, i.e., the average of the integrated function over Ω. We use function space notation such as $L^2(\Omega;\mathbb{R}^3)$,$H^1_0(\mathbb{R}^3;\mathbb{R}^3)$, $H^1(V;\mathbb{R}^3)$, see Rudin (1987) and Adams (1978) for details. While the arguments given below are rigorous, our focus is on the physical implications of the constrained theory and Eshelby's solution, and the prediction of properties, so our presentation can be read without a precise knowledge of these spaces.

2. The constrained theory for an isolated magnetostrictive particle

We use the large-body limit of micromagnetics to describe the behavior of the magnetostrictive particles. Micromagnetics was first developed by Brown (1963), based on earlier more physical ideas of Landau and Lifshitz (1935) and Lifshitz (1944), and the large body limit of micromagnetics was derived by DeSimone (1993). In simple terms the magnetic body is assumed to be large enough that the exchange energy can be neglected, and one needs to study the properties of minimizing sequences, as the energy may not have a minimizer. In this large body limit the free energy of an isolated single crystal magnetostrictive particle under the application of an external magnetic field $\mathbf{h}_e$ and stress $\mathbf{S}_e$ is written as

$$\mathscr{E}^*(\mathbf{u}(\mathbf{x}),\mathbf{m}(\mathbf{x})) = \int_\Omega [\Phi(\mathbf{E}[\mathbf{u}](\mathbf{x}),\mathbf{m}(\mathbf{x})) - \mathbf{h}_e\cdot\mathbf{m}(\mathbf{x}) - \mathbf{S}_e\cdot\mathbf{E}[\mathbf{u}](\mathbf{x})]\,d\mathbf{x} + \frac{1}{2\gamma}\int_{\mathbb{R}^3} |\nabla\xi_{\mathbf{m}}(\mathbf{x})|^2\,d\mathbf{x}, \tag{1}$$

where Ω is the reference configuration of the particle, $(\mathbf{u}(\mathbf{x}),\mathbf{m}(\mathbf{x}))$ are the displacement and magnetization defined on Ω, and $\mathbf{E}[\mathbf{u}] = \frac{1}{2}(\nabla\mathbf{u} + (\nabla\mathbf{u})^{\mathrm{T}})$ is the strain. Since the material is ferromagnetic, the magnitude of the magnetization must be almost everywhere a constant, the *saturation magnetization*, denoted by m_s. The function $\Phi(\mathbf{E},\mathbf{m}) : \mathbb{R}^{3\times3}_{\mathrm{sym}}\times\mathbb{R}^3 \mapsto \mathbb{R}$ is the anisotropy energy density, and reflects the fact that the crystal prefers certain magnetization directions (easy axes) and the associated stress-free strains. The term $\frac{1}{2\gamma}\int_{\mathbf{R}^3}|\nabla\xi_{\mathbf{m}}(\mathbf{x})|^2\,d\mathbf{x}$ is the magnetostatic energy. The magnetic potential $\xi_{\mathbf{m}}(\mathbf{x})$, arising from the magnetization of the particle itself, is determined by the Maxwell equation

$$\mathrm{div}(-\nabla\xi_{\mathbf{m}}(\mathbf{x}) + \gamma\mathbf{m}(\mathbf{x})\chi_\Omega(\mathbf{x})) = 0 \quad \text{in } \mathbb{R}^3, \tag{2}$$

where γ is a constant depending on the system of units, and $\chi_\Omega : \mathbb{R}^3 \mapsto \mathbb{R}$, the characteristic function of region Ω, takes values 1 inside Ω and 0 elsewhere. Finally, the term $\mathbf{h}_e\cdot\mathbf{m}(\mathbf{x}) + \mathbf{S}_e\cdot\mathbf{E}(\mathbf{x})$ gives the potential energy density from the applied magnetic field and stress, where both $\mathbf{h}_e \in \mathbb{R}^3$ and $\mathbf{S}_e \in \mathbb{R}^{3\times3}_{\mathrm{sym}}$ are assumed to be constant. At a fixed temperature below the *Curie temperature*, the magnetoelastic properties of the particle are governed by the minimization problem (see DeSimone, 1993)

$$\min_{(\mathbf{u},\mathbf{m})\in\mathscr{A}^*} \mathscr{E}^*(\mathbf{u}(\mathbf{x}),\mathbf{m}(\mathbf{x})), \tag{3}$$

where the admissible space

$$\mathscr{A}^* \equiv \{(\mathbf{u},\mathbf{m}) \in H^1(\Omega;\mathbb{R}^3)\times L^2(\Omega;\mathbb{R}^3) \,|\, |\mathbf{m}(\mathbf{x})| = m_s \text{ a.e. in } \Omega\}. \tag{4}$$

We now consider the free energy (1) in the context of constrained theory. We define the energy wells as pairs of strain-magnetization $(\mathbf{E},\mathbf{m}) \in \mathbb{R}^{3\times 3}_{\text{sym}} \times \mathbb{R}^3$ that minimize the anisotropy energy density $\Phi(\mathbf{E},\mathbf{m})$. Without loss of generality, one may assume the minimum is zero. In general, there are an even and finite number of energy-wells due to the underlying crystalline structure, see James and Wuttig (1998); James and Hane (2000). Thus,

$$\Phi(\mathbf{E},\mathbf{m}) \geqslant 0, \quad \Phi(\mathbf{E},\mathbf{m}) = 0 \iff (\mathbf{E},\mathbf{m}) \in \mathbb{K} \equiv \bigcup_{i=1}^{n} (\mathbf{E}_i,\mathbf{m}_i), \tag{5}$$

where $\mathbb{K}$ is the collection of all energy-wells. A variant refers to a local region inside the material with strain-magnetization pair falling on exactly one of the energy-wells. Therefore there are n variants, where n is the number of energy-wells.[1]

We are only interested in magnetostrictive materials with the following special features: high anisotropy (magnetically and elastically hard) and pairwise compatible, mobile variant interfaces. By high anisotropy, we mean

$$\frac{|\Phi(\mathbf{E},\mathbf{m})|}{m_s^2} \gg 1 \quad \text{for a typical strain-magnetization pair } (\mathbf{E},\mathbf{m}) \notin \mathbb{K}. \tag{6}$$

Pairwise compatible interfaces are defined such that for any two different energy-wells $(\mathbf{E}_j,\mathbf{m}_j),(\mathbf{E}_l,\mathbf{m}_l) \in \mathbb{K}$, there exist a unit vector $\mathbf{n}_{jl}$ and a vector $\mathbf{a}_{jl}$ such that

$$\begin{cases} \mathbf{E}_j - \mathbf{E}_l = \frac{1}{2}(\mathbf{a}_{jl} \otimes \mathbf{n}_{jl} + \mathbf{n}_{jl} \otimes \mathbf{a}_{jl}) \\ (\mathbf{m}_j - \mathbf{m}_l) \cdot \mathbf{n}_{jl} = 0 \end{cases} \quad \text{(no summation)}, \tag{7}$$

where the first equation in (7) is the Hadamard jump condition and the second equation is the magnetic compatibility condition. Therefore, an interface with normal $\mathbf{n}_{jl}$, separating the jth variant and lth variant, satisfies the required continuity of the displacement and is also pole-free, i.e., does not by itself induce magnetostatic energy. These conditions are satisfied among all the common giant magnetostrictive and ferromagnetic shape memory materials. Finally if the interfaces are mobile, the use of a minimum energy principle for the energy (1) is justified.

To exploit the consequences of these special features of the magnetostrictive material, a constrained theory was proposed by DeSimone and James (2002). In this theory, one introduces a single dimensionless scaling parameter ε such that

$$\varepsilon\Phi(\mathbf{E},\mathbf{m}) = \Phi_1(\mathbf{E},\mathbf{m}) \sim m_s^2,$$

and rewrites the free energy (1) as

$$\begin{aligned} \mathcal{E}^*_\varepsilon(\mathbf{u},\mathbf{m}) = & \int_\Omega \left[\frac{1}{\varepsilon}\Phi_1(\mathbf{E}[\mathbf{u}](\mathbf{x}),\mathbf{m}(\mathbf{x})) - \mathbf{h}_e \cdot \mathbf{m}(\mathbf{x}) - \mathbf{S}_e \cdot \mathbf{E}[\mathbf{u}](\mathbf{x})\right] \mathrm{d}\mathbf{x} \\ & + \frac{1}{2\gamma}\int_{\mathbb{R}^3} |\nabla \xi_{\mathbf{m}}(\mathbf{x})|^2 \,\mathrm{d}\mathbf{x}. \end{aligned} \tag{8}$$

From Eq. (6), we expect ε is a very small positive number for highly anisotropic materials. However, under a typical applied magnetic field of the order γm_s and moderate applied

[1] In our terminology pairs of wells of the form $(\mathbf{E}, \pm\mathbf{m})$ are considered different variants.

stress, the terms

$$\frac{1}{|\Omega|}\left[\int_\Omega -\mathbf{h}_e\cdot\mathbf{m}(\mathbf{x})-\mathbf{S}_e\cdot\mathbf{E}[\mathbf{u}](\mathbf{x})\,d\mathbf{x}+\frac{1}{2\gamma}\int_{\mathbb{R}^3}|\nabla\xi_{\mathbf{m}}(\mathbf{x})|^2\,d\mathbf{x}\right]\sim m_e^2\quad\forall(\mathbf{u},\mathbf{m})\in\mathscr{A}^*,\tag{9}$$

where $|\Omega|$ is the volume of Ω. Thus, one can argue that the effective properties governed by minimization problem (3) are well predicted by the asymptotic properties governed by minimization problems $\min_{\mathscr{A}^*}\mathscr{E}_\varepsilon^*(\mathbf{u},\mathbf{m})$ as ε tends to 0. This leads to studying the Γ-limit of the energy functionals $\mathscr{E}_\varepsilon^*$, see Dal Maso (1993). Very similar ideas have been employed by Bhattacharya and Li (2001) to predict behaviors of ferroelectrics. In effect, DeSimone and James (2002) have already proved the following theorem:

Theorem 2.1. *Define the (relaxed) energy functional* $\mathscr{E}_0^*:\mathscr{A}_0^*\mapsto\mathbb{R}$ *as*

$$\mathscr{E}_0^*(\mathbf{u},\mathbf{m})\equiv\int_\Omega[-\mathbf{h}_e\cdot\mathbf{m}(\mathbf{x})-\mathbf{S}_e\cdot\mathbf{E}[\mathbf{u}](\mathbf{x})]\,d\mathbf{x}+\frac{1}{2\gamma}\int_{\mathbb{R}^3}|\nabla\xi_{\mathbf{m}}(\mathbf{x})|^2\,d\mathbf{x},$$

and the function space

$$\mathscr{A}_0^*\equiv\{(\mathbf{u},\mathbf{m})\in H^1(\Omega;\mathbb{R}^3)\times L^2(\Omega;\mathbb{R}^3)\,|\,(\mathbf{E}[\mathbf{u}](\mathbf{x}),\mathbf{m}(\mathbf{x}))\in\mathrm{Co}(\mathbb{K})\text{ a.e. in }\Omega\},$$

where $\mathrm{Co}(\mathbb{K})\subset\mathbb{R}^{3\times3}_{\mathrm{sym}}\times\mathbb{R}^3$ *denotes the convex hull of the energy-wells* $\mathbb{K}$

$$\mathrm{Co}(\mathbb{K})\equiv\left\{(\mathbf{E},\mathbf{m})\,|\,(\mathbf{E},\mathbf{m})=\sum_{i=1}^n\theta_i(\mathbf{E}_i,\mathbf{m}_i),0\leqslant\theta_i\leqslant1,\ \sum_{i=1}^n\theta_i=1,(\mathbf{E}_i,\mathbf{m}_i)\in\mathbb{K}\ \forall i=1,\dots,n\right\}.\tag{10}$$

If the energy-wells in $\mathbb{K}$ *are pairwise compatible, then we have convergence of infimums*

$$\liminf_{\varepsilon\to0}\ \inf_{\mathscr{A}^*}\mathscr{E}_\varepsilon^*=\inf_{\mathscr{A}_0^*}\mathscr{E}_0^*,$$

and also convergence of minimizers. That is, if $(\mathbf{u}_0(\mathbf{x}),\mathbf{m}_0(\mathbf{x}))\in\mathscr{A}_0^*$ *is the unique minimizer of* $\mathscr{E}_0^*$*, then the low-energy sequence* $(\mathbf{u}_\varepsilon(\mathbf{x}),\mathbf{m}_\varepsilon(\mathbf{x}))\in\mathscr{A}^*$ *of* $\mathscr{E}_\varepsilon^*$*, defined as*

$$\mathscr{E}_\varepsilon^*(\mathbf{u}_\varepsilon(\mathbf{x}),\mathbf{m}_\varepsilon(\mathbf{x}))\leqslant\inf_{\mathscr{A}^*}\mathscr{E}_\varepsilon^*+\varepsilon\quad\forall\varepsilon>0,$$

satisfies (as $\varepsilon\to0$*),*

$$(\mathbf{u}_\varepsilon(\mathbf{x}),\mathbf{m}_\varepsilon(\mathbf{x}))\rightharpoonup(\mathbf{u}_0(\mathbf{x}),\mathbf{m}_0(\mathbf{x}))\in\mathscr{A}_0^*\quad\text{weakly in }H^1(\Omega;\mathbb{R}^3)\times L^2(\Omega;\mathbb{R}^3)\tag{11}$$

and

$$\liminf_{\varepsilon\to0}\ \mathscr{E}^*(\mathbf{u}_\varepsilon(\mathbf{x}),\mathbf{m}_\varepsilon(\mathbf{x}))=\mathscr{E}_0^*(\mathbf{u}_0(\mathbf{x}),\mathbf{m}_0(\mathbf{x}))=\inf_{(\mathbf{u},\mathbf{m})\in\mathscr{A}_0^*}\mathscr{E}_0^*(\mathbf{u},\mathbf{m}).\tag{12}$$

From Eqs. (11) to (12), we conclude that the (relaxed) minimization problem

$$\min_{(\mathbf{u},\mathbf{m})\in\mathscr{A}_0^*}\mathscr{E}_0^*(\mathbf{u}(\mathbf{x}),\mathbf{m}(\mathbf{x}))\tag{13}$$

can effectively predict the properties, such as local average strain and local average magnetization, of an isolated single-crystal magnetostrictive particle having high anisotropy energy and mobile variant interfaces.

3. Magnetostrictive composites in the dilute limit

3.1. The constrained theory for the composite system

The composite under consideration consists of well separated identical magnetostrictive particles surrounded by an elastic matrix. It is assumed that the matrix is magnetically transparent and obeys linear elasticity, with elasticity tensor $\mathbf{C}$, and that the magnetostrictive material has high anisotropy energy and mobile variant interfaces as described in the last section. Note that the anisotropy energy includes the elastic energy of the inclusion, so the assumption that the magnetostrictive material has high anisotropy energy implies that the matrix should be much softer than a single variant of the magnetostrictive phase. In the dilute limit, the magnetostrictive particles inside the composite are far apart and hence the magnetic and elastic interactions between different particles are negligible, see Christensen (1979). The physical idea here is that each particle has a certain "sphere of influence" whose properties are substantially unaffected by the other particles. Quantities like the strain of the composite are calculated as averages over this sphere of influence, and, of course, this average strain goes to zero (if there is no long range stress) as this sphere of influence gets larger and larger. However, the average strain times the volume of the sphere of influence tends to a definite quantity, which has the interpretation as the first approximation to the average strain in a small-volume-fraction expansion; and this is what is plotted in the results. These ideas can be made rigorous. Using these ideas, it is appropriate to model the composite system by a single magnetostrictive inclusion embedded in an infinite linear elastic matrix, see Fig. 1.

Let Ω be the region occupied by the inclusion of magnetostrictive material, $(\mathbf{u}(\mathbf{x}), \mathbf{m}(\mathbf{x}))$ be the displacement and magnetization defined on $\mathbb{R}^3$ and Ω, respectively. Further, it is assumed that the potential energy due to an external mechanical loading device can be written as

$$-\int_{\Omega} \mathbf{S}_e \cdot \mathbf{E}[\mathbf{u}](\mathbf{x})\,d\mathbf{x}, \tag{14}$$

where $\mathbf{S}_e \in \mathbb{R}^{3\times 3}_{\mathrm{sym}}$ is constant. Note that the integral in (14) is only over the inclusion; in the next section we give the requirements on the loading device such that it can be described by

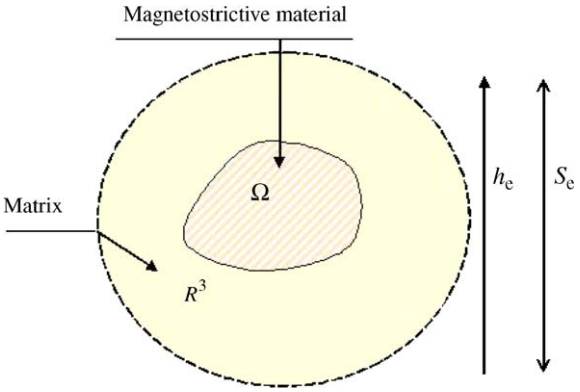

Fig. 1. Modelling a magnetostrictive composite system in the dilute limit.

an "effective" stress $\mathbf{S}_e$ as in Eq. (14) and we give a formula that relates $\mathbf{S}_e$ to the actual applied loads. Thus, under the application of the uniform magnetic field $\mathbf{h}_e$ and this loading device, the free energy of this composite model in the constrained theory can be written as

$$\mathscr{E}_0(\mathbf{u}(\mathbf{x}),\mathbf{m}(\mathbf{x})) = \int_{\mathbb{R}^3}\left[\tfrac{1}{2}(1-\chi_\Omega)\nabla\mathbf{u}\cdot\mathbf{C}\nabla\mathbf{u}+\frac{1}{2\gamma}|\nabla\xi_{\mathbf{m}}|^2\right]\mathrm{d}\mathbf{x} - \int_\Omega(\mathbf{S}_e\cdot\mathbf{E}[\mathbf{u}]+\mathbf{h}_e\cdot\mathbf{m})\,\mathrm{d}\mathbf{x}, \tag{15}$$

and properties such as local average strain and local average magnetization are well predicted by the (relaxed) minimization problem

$$\min_{(\mathbf{u},\mathbf{m})\in\mathscr{A}_0}\mathscr{E}_0(\mathbf{u}(\mathbf{x}),\mathbf{m}(\mathbf{x})), \tag{16}$$

where the function space $\mathscr{A}_0$ is defined by

$$\mathscr{A}_0 \equiv \{(\mathbf{u},\mathbf{m})\in H_0^1(\mathbb{R}^3;\mathbb{R}^3)\times L^2(\Omega;\mathbb{R}^3)\,|\,(\mathbf{E}[\mathbf{u}](\mathbf{x}),\mathbf{m}(\mathbf{x}))\in \mathrm{Co}(\mathbb{K}) \text{ a.e. in } \Omega\}. \tag{17}$$

Note that the condition $\mathbf{u}\in H_0^1(\mathbb{R}^3;\mathbb{R}^3)$ means $\int_{\mathbb{R}^3}[|\mathbf{u}|^2+|\nabla\mathbf{u}|^2]\,\mathrm{d}\mathbf{x}<+\infty$, and so implies that $\mathbf{u}(\mathbf{x})$ and $\nabla\mathbf{u}(\mathbf{x})$ decay faster than $1/|\mathbf{x}|^{3/2}$ as $|\mathbf{x}|\to+\infty$.

3.2. Ellipsoidal inclusions

The minimization problem (16), like the minimization problem (3), can be further simplified if the inclusion Ω is an ellipsoid. To see this, we compare the free energy of a given state $(\mathbf{u}(\mathbf{x}),\mathbf{m}(\mathbf{x}))\in\mathscr{A}_0$ with the state $(\bar{\mathbf{u}}(\mathbf{x}),\bar{\mathbf{m}}(\mathbf{x}))\in\mathscr{A}_0$ defined by

$$\begin{cases}\mathrm{div}(\mathbf{C}\nabla\bar{\mathbf{u}}(\mathbf{x}))=0 & \forall\mathbf{x}\in\mathbb{R}^3\backslash\Omega\\ \bar{\mathbf{u}}(\mathbf{x})=\mathbf{F}^0\mathbf{x} & \forall\mathbf{x}\in\Omega\end{cases}\quad\text{and}\quad \bar{\mathbf{m}}(\mathbf{x})=\chi_\Omega(\mathbf{x})\mathbf{m}^0, \tag{18}$$

where $\mathbf{F}^0=\fint_\Omega\nabla\mathbf{u}(\mathbf{t})\,\mathrm{d}\mathbf{t}$ and $\mathbf{m}^0=\fint_\Omega\mathbf{m}(\mathbf{t})\,\mathrm{d}\mathbf{t}$. It is clear that $(\bar{\mathbf{u}}(\mathbf{x}),\bar{\mathbf{m}}(\mathbf{x}))$ are well-defined since the equations used to define $\bar{\mathbf{u}}(\mathbf{x})$ have a unique solution in the space $H_0^1(\mathbb{R}^3;\mathbb{R}^3)$, see Evans (1998, p. 293). It is also easy to verify that the linear term in $\mathscr{E}_0$, i.e., the last integral in Eq. (15) is the same for the states $(\mathbf{u}(\mathbf{x}),\mathbf{m}(\mathbf{x}))$ and $(\bar{\mathbf{u}}(\mathbf{x}),\bar{\mathbf{m}}(\mathbf{x}))$. It has been shown in DeSimone and James (2002) that a nonuniform magnetization on an ellipsoidal region induces higher magnetostatic energy than the uniform magnetization with the same average. That is, we have the following inequality if Ω is an ellipsoid

$$\int_{\mathbb{R}^3}|\nabla\xi_{\mathbf{m}}(\mathbf{x})|^2\,\mathrm{d}\mathbf{x}\geqslant\int_{\mathbb{R}^3}|\nabla\xi_{\bar{\mathbf{m}}}(\mathbf{x})|^2\,\mathrm{d}\mathbf{x}=|\Omega|\gamma\mathbf{m}^0\cdot\mathbf{D}\mathbf{m}^0, \tag{19}$$

where $\xi_{\bar{\mathbf{m}}}(\mathbf{x})$ is determined by Eq. (2) with $\mathbf{m}(\mathbf{x})$ replaced by $\bar{\mathbf{m}}(\mathbf{x})$, and the *demagnetization matrix* $\mathbf{D}$ depends only the shape of the ellipsoid Ω (demagnetization matrices are tabulated). Further, we can show that

$$\int_{\mathbb{R}^3}(1-\chi_\Omega)\nabla\mathbf{u}(\mathbf{x})\cdot\mathbf{C}\nabla\mathbf{u}(\mathbf{x})\,\mathrm{d}\mathbf{x}\geqslant\int_{\mathbb{R}^3}(1-\chi_\Omega)\nabla\bar{\mathbf{u}}(\mathbf{x})\cdot\mathbf{C}\nabla\bar{\mathbf{u}}(\mathbf{x})\,\mathrm{d}\mathbf{x} = |\Omega|\mathbf{F}^0\cdot(\mathbf{R}^{-1}-\mathbf{C})\mathbf{F}^0 \tag{20}$$

by noticing that the surface traction on $\partial\Omega^+$ determined by Eq. (18) can be expressed as (see the Appendix)

$$(\mathbf{C}\nabla\bar{\mathbf{u}}(\mathbf{x}))\mathbf{n} = ((\mathbf{C}-\mathbf{R}^{-1})\mathbf{F}^0)\mathbf{n} \quad \forall \mathbf{x} \in \partial\Omega^+ \tag{21}$$

for an ellipsoidal region Ω, where $\mathbf{n}$ is the outward normal on $\partial\Omega$, $\partial\Omega^+$ means the limit approached from the outside of Ω, and $\mathbf{R} \in \mathbb{R}^{3\times3\times3\times3}$ depends on the ellipsoidal region Ω and the elasticity tensor $\mathbf{C}$. See the Appendix for the explicit form of the tensor $\mathbf{R}$ and its relation to the classical *Eshelby tensor*. The proof of inequality (20) is as follows:

From the divergence theorem and Eq. (18), we have

$$\int_{\mathbb{R}^3\setminus\Omega} \nabla\mathbf{v}\cdot\mathbf{C}\nabla\bar{\mathbf{u}}\,\mathrm{d}\mathbf{x} = -\int_{\partial\Omega^+} \mathbf{n}\otimes\mathbf{v}\cdot\mathbf{C}\nabla\bar{\mathbf{u}}\,\mathrm{d}\mathbf{x} \quad \forall \mathbf{v} \in H_0^1(\mathbb{R}^3;\mathbb{R}^3). \tag{22}$$

Thus,

$$\begin{aligned}\int_{\mathbb{R}^3\setminus\Omega} \nabla\mathbf{u}\cdot\mathbf{C}\nabla\mathbf{u}\,\mathrm{d}\mathbf{x} &= \int_{\mathbb{R}^3\setminus\Omega} [\nabla\bar{\mathbf{u}}\cdot\mathbf{C}\nabla\bar{\mathbf{u}} + 2\nabla\bar{\mathbf{u}}\cdot\mathbf{C}\nabla(\mathbf{u}-\bar{\mathbf{u}}) + \nabla(\mathbf{u}-\bar{\mathbf{u}})\cdot\mathbf{C}\nabla(\mathbf{u}-\bar{\mathbf{u}})]\,\mathrm{d}\mathbf{x}\\ &= \int_{\mathbb{R}^3\setminus\Omega} \nabla\bar{\mathbf{u}}\cdot\mathbf{C}\nabla\bar{\mathbf{u}}\,\mathrm{d}\mathbf{x} + \int_{\mathbb{R}^3\setminus\Omega} \nabla(\mathbf{u}-\bar{\mathbf{u}})\cdot\mathbf{C}\nabla(\mathbf{u}-\bar{\mathbf{u}})\,\mathrm{d}\mathbf{x}\\ &\quad - 2\int_{\partial\Omega^+} \mathbf{n}\otimes(\mathbf{u}-\bar{\mathbf{u}})\cdot\mathbf{C}\nabla\bar{\mathbf{u}}\,\mathrm{d}\mathbf{x}.\end{aligned}$$

Since Ω is an ellipsoid, from Eq. (21) and definition (18) of $\bar{\mathbf{u}}$, we have

$$\begin{aligned}\int_{\partial\Omega^+} \mathbf{n}\otimes(\mathbf{u}-\bar{\mathbf{u}})\cdot\mathbf{C}\nabla\bar{\mathbf{u}}\,\mathrm{d}\mathbf{x} &= \int_{\partial\Omega^+} \mathbf{n}\otimes(\mathbf{u}-\bar{\mathbf{u}})\cdot(\mathbf{C}-\mathbf{R}^{-1})\mathbf{F}^0\,\mathrm{d}\mathbf{x}\\ &= \left[\int_{\partial\Omega^+} \mathbf{n}\otimes(\mathbf{u}-\bar{\mathbf{u}})\,\mathrm{d}\mathbf{x}\right]\cdot(\mathbf{C}-\mathbf{R}^{-1})\mathbf{F}^0 = 0,\end{aligned}$$

and hence inequality (20) follows from the positive-definiteness of $\mathbf{C}$.

We conclude that the minimizer of the minimization problem (16) must be of the form (18) for an ellipsoidal region Ω. It is clear that for these states the energy $\mathscr{E}_0$ defined in (15) is completely determined by the average deformation $\mathbf{F}^0$ and average magnetization $\mathbf{m}^0$, i.e.,

$$\mathscr{E}_0(\bar{\mathbf{u}}(\mathbf{x}),\bar{\mathbf{m}}(\mathbf{x})) = |\Omega|\mathscr{E}_0^\star(\mathbf{F}^0,\mathbf{m}^0),$$

where from Eqs. (19) to (20),

$$\mathscr{E}_0^\star(\mathbf{F},\mathbf{m}) \equiv \tfrac{1}{2}\mathbf{F}\cdot(\mathbf{R}^{-1}-\mathbf{C})\mathbf{F} + \tfrac{1}{2}\mathbf{m}\cdot\mathbf{D}\mathbf{m} - \mathbf{S}_\mathrm{e}\cdot\mathbf{F} - \mathbf{h}_\mathrm{e}\cdot\mathbf{m}. \tag{23}$$

In the Appendix, we show that the function $\mathscr{E}_0^\star$ can be rewritten as (see Eq. (57))

$$\mathscr{E}_0^\star(\mathbf{F},\mathbf{m}) = \mathscr{E}_{0E}^\star(\mathbf{E},\mathbf{m}) + \mathscr{E}_{0W}^\star(\mathbf{W}) \tag{24}$$

where $\mathbf{E} = \frac{1}{2}(\mathbf{F}+\mathbf{F}^\mathrm{T})$ is symmetric, $\mathbf{W} = \mathbf{F} - \mathbf{R}\mathbf{C}\mathbf{S}^{-1}\mathbf{E}$ is skew-symmetric, and $\mathbf{S} \in \mathbb{R}^{3\times3\times3\times3}$ is the *Eshelby tensor*, see Eshelby (1957) and Mura (1987). Also the functions $\mathscr{E}_{0E}^\star$ and $\mathscr{E}_{0W}^\star$ are

$$\begin{aligned}\mathscr{E}_{0E}^\star(\mathbf{E},\mathbf{m}) &= \tfrac{1}{2}\mathbf{E}\cdot(\mathbf{C}\mathbf{S}^{-1}-\mathbf{C})\mathbf{E} + \tfrac{1}{2}\mathbf{m}\cdot\mathbf{D}\mathbf{m} - \mathbf{S}_\mathrm{e}\cdot\mathbf{E} - \mathbf{h}_\mathrm{e}\cdot\mathbf{m},\\ \mathscr{E}_{0W}^\star(\mathbf{W}) &= \tfrac{1}{2}\mathbf{W}\cdot\mathbf{R}^{-1}\mathbf{W},\end{aligned}$$

respectively.

Let $(\mathbf{F}^\star, \mathbf{m}^\star)$ be the corresponding average deformation and magnetization of the minimizer of the minimization problem (16), i.e.,

$$|\Omega|\mathscr{E}_0^\star(\mathbf{F}^\star, \mathbf{m}^\star) = \mathscr{E}_0(\bar{\mathbf{u}}^\star(\mathbf{x}), \bar{\mathbf{m}}^\star(\mathbf{x})) = \min_{(\mathbf{u},\mathbf{m}) \in \mathscr{A}_0} \mathscr{E}_0(\mathbf{u}(\mathbf{x}), \mathbf{m}(\mathbf{x})),$$

then from Eqs. (16) to (17), we have

$$\mathscr{E}_0^\star(\mathbf{F}^\star, \mathbf{m}^\star) = \min_{(\mathbf{E},\mathbf{m}) \in \mathrm{Co}(\mathbb{K})} \mathscr{E}_0^\star(\mathbf{F}, \mathbf{m}). \tag{25}$$

From Eq. (20) one knows $\mathbf{R}^{-1} - \mathbf{C}$ is positive-definite. It is also easy to verify that the admissible space for $(\mathbf{F}, \mathbf{m})$ is convex with finite dimensions. Thus the minimization problem (25) is a positive-definite quadratic programming problem and has a unique minimizer $(\mathbf{F}^\star, \mathbf{m}^\star)$. Further, from Eq. (24) and positive-definiteness of $\mathbf{R}^{-1}$, one can rewrite the minimization problem (25) as

$$\begin{aligned} \mathscr{E}_0^\star(\mathbf{F}^\star, \mathbf{m}^\star) &= \min_{(\mathbf{E},\mathbf{m}) \in \mathrm{Co}(\mathbb{K})} \left[\mathscr{E}_{0E}^\star(\mathbf{E}, \mathbf{m}) + \min_{\mathbf{W} \in \mathbb{R}^{3\times 3}_{\mathrm{skew}}} \mathscr{E}_{0W}^\star(\mathbf{W}) \right] \\ &= \min_{(\mathbf{E},\mathbf{m}) \in \mathrm{Co}(\mathbb{K})} \mathscr{E}_{0E}^\star(\mathbf{E}, \mathbf{m}). \end{aligned} \tag{26}$$

In terms of volume fractions Θ of energy-wells, the quadratic programming problem (26) can be written as

$$Q(\Theta^\star) = \min_{\Theta \in \mathbb{F}} \left\{ Q(\Theta) \equiv \tfrac{1}{2}\Theta \cdot \mathbf{A}\Theta - \mathbf{b} \cdot \Theta \right\}, \tag{27}$$

where $\Theta = (\theta_1, \ldots, \theta_n) \in \mathbb{R}^n$ are the volume fractions of the n energy-wells in $\mathbb{K}$ with

$$\mathbb{F} = \left\{ \Theta \,\middle|\, \sum_{i=1}^{n} \theta_i = 1, \theta_i \in [0, 1] \ \forall i = 1, \ldots, n \right\}.$$

Also $\mathbf{A}$ and $\mathbf{b}$, a constant $n \times n$ matrix and a $1 \times n$ vector, are given by

$$\begin{cases} \mathbf{A} = (A_{ij}) = (\mathbf{m}_i \cdot \mathbf{D}\mathbf{m}_j + \mathbf{E}_i \cdot (\mathbf{C}\mathbf{S}^{-1} - \mathbf{C})\mathbf{E}_j), & i,j = 1, \ldots, n, \\ \mathbf{b} = (b_i) = (\mathbf{h}_\mathrm{e} \cdot \mathbf{m}_i + \mathbf{S}_\mathrm{e} \cdot \mathbf{E}_i), & i = 1, \ldots, n, \end{cases} \tag{28}$$

where $(\mathbf{E}_i, \mathbf{m}_i) \in \mathbb{K}\ \forall i = 1, \ldots, n$ are the energy-wells. Clearly, $(\mathbf{F}^\star, \mathbf{m}^\star)$ is related to $\Theta^\star$ by (see Eq. (26) and the identity $\mathbf{W} = \mathbf{F} - \mathbf{R}\mathbf{C}\mathbf{S}^{-1}\mathbf{E}$)

$$(\mathbf{E}^\star, \mathbf{m}^\star) = \sum_{i=1}^{n} \theta_i^\star(\mathbf{E}_i, \mathbf{m}_i) \quad \text{and} \quad \mathbf{F}^\star = \mathbf{R}\mathbf{C}\mathbf{S}^{-1}\mathbf{E}^\star. \tag{29}$$

In summary, for the composite system under consideration, the average strain and magnetization inside the magnetostrictive inclusion are well predicted by the quadratic programming problem (26) under the assumptions: (1) the magnetostrictive material has the special features, i.e., high anisotropy and mobile variant interfaces, as in the original constrained theory; (2) the matrix obeys linear elasticity, is magnetically transparent, and is much softer than each magnetostrictive variant; (3) the inclusion region Ω is an ellipsoid.

To understand better the relative importance of the various terms of the energy, it is useful to develop some simple bounds. A lower bound on the energy is obtained by neglecting the magnetostatic energy, while an upper bound is obtained by restricting the strain and magnetization to be uniform on the inclusion. That is,

$$\min_{(\mathbf{u},\mathbf{m}) \in \mathscr{A}_0} \mathscr{E}_l(\mathbf{u}(\mathbf{x}), \mathbf{m}(\mathbf{x})) \leqslant \min_{(\mathbf{u},\mathbf{m}) \in \mathscr{A}_0} \mathscr{E}_0(\mathbf{u}(\mathbf{x}), \mathbf{m}(\mathbf{x})) \leqslant \min_{(\mathbf{u},\mathbf{m}) \in \mathscr{A}_u} \mathscr{E}_0(\mathbf{u}(\mathbf{x}), \mathbf{m}(\mathbf{x})), \tag{30}$$

where

$$\mathscr{E}_l(\mathbf{u},\mathbf{m}) \equiv \int_\Omega [-\mathbf{h}_e \cdot \mathbf{m}(\mathbf{x}) - \mathbf{S}_e \cdot \mathbf{E}[\mathbf{u}](\mathbf{x})]\,d\mathbf{x} + \int_{\mathbb{R}^3} \tfrac{1}{2}(1-\chi_\Omega)\nabla\mathbf{u}\cdot\mathbf{C}\nabla\mathbf{u}\,d\mathbf{x},$$

and the function space

$$\mathscr{A}_u \equiv \{(\mathbf{u},\mathbf{m}) \in \mathscr{A}_0 \,|\, (\mathbf{E}[\mathbf{u}](\mathbf{x}),\mathbf{m}(\mathbf{x})) \text{ are uniform inside } \Omega\}.$$

In the case of an ellipsoidal inclusion, the upper bound matches the exact energy, because of the special properties of ellipsoids used throughout this paper.

The lower bound is examined in two interesting cases in Fig. 3. On the left of Fig. 3, we show the predicted response of Ni_2MnGa compared to the lower bound. It is seen that the presence of magnetostatic energy has a moderate effect on the transition fields. For Terfenol-D the effect of magnetostatic energy is much more dramatic; in its absence the fields at transition drop to a small fraction of their predicted values. The shape of the response curve of the lower bound is similar to the predicted response, but on the scale of Fig. 3 it appears to collapse to a horizontal line at the saturation strain with a single point at $h_e = 0$ and zero strain.

4. Magnetoelastic properties of the composite system in the dilute limit

We now calculate the average strain of the composite system in terms of the average strain of the magnetostrictive inclusion, which is predicted by the quadratic programming problem (26). Consider a composite specimen, let V be the region occupied by the whole composite specimen, $\Omega \subset V$ be the regions occupied by the magnetostrictive particles. In the dilute limit, interactions between different particles are negligible. Thus, we may assume there is only one magnetostrictive particle embedded in V, as sketched in Fig. 2. A mechanical loading device can be modelled by some surface traction $\mathbf{t}(\mathbf{x}) : \partial V \mapsto \mathbb{R}^3$. Then, the displacement restricted to the matrix satisfies the following equations:

$$\begin{cases} \operatorname{div}[\mathbf{C}\nabla\mathbf{u}(\mathbf{x})] = 0, & \mathbf{x} \in V\backslash\Omega, \\ (\mathbf{C}\nabla\mathbf{u}(\mathbf{x}))\mathbf{n}(\mathbf{x}) = \mathbf{t}(\mathbf{x}), & \mathbf{x} \in \partial V. \end{cases} \tag{31}$$

The displacement restricted to the inclusion, determined by the minimization problem (26) in the dilute limit and for ellipsoidal Ω, can be written as

$$\mathbf{u}(\mathbf{x}) = \mathbf{F}^\star\mathbf{x} \quad \mathbf{x} \in \Omega.$$

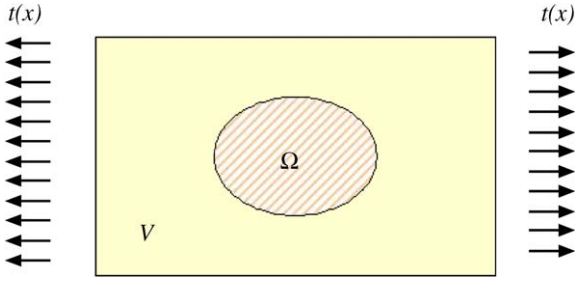

Fig. 2. A composite specimen under the application of a dead load.

Our first task is to obtain a simple decomposition for the overall strain of the composite. We first observe that there is a part of the overall strain that is a simple linear transformation of $\mathbf{E}^\star = \frac{1}{2}(\mathbf{F}^\star + (\mathbf{F}^\star)^{\mathrm{T}})$. To see this, we calculate the overall strain relative to a certain reference state $\mathbf{u}_0$:

$$\mathbf{E}^C = \frac{1}{|V|}\int_V \mathbf{E}[\mathbf{u}-\mathbf{u}_0](\mathbf{x})\,\mathrm{d}\mathbf{x}.$$

Here, the reference displacement $\mathbf{u}_0$ can be chosen to be the displacement, under the influence of the same loads, that would be present if the inclusion were rigidified:

$$\begin{cases} \operatorname{div}[\mathbf{C}\nabla\mathbf{u}_0(\mathbf{x})] = 0, & \mathbf{x}\in V\backslash\Omega,\\ \mathbf{u}_0(\mathbf{x}) = 0, & \mathbf{x}\in\Omega,\\ (\mathbf{C}\nabla\mathbf{u}_0(\mathbf{x}))\mathbf{n}(\mathbf{x}) = \mathbf{t}(\mathbf{x}), & \mathbf{x}\in\partial V. \end{cases} \tag{32}$$

The displacement $\mathbf{u}_0$ is well-defined since the above equations have a unique solution in $H^1(V;\mathbb{R}^3)$. By the divergence theorem, we have

$$\begin{aligned} \mathbf{E}^C &= \frac{1}{|V|}\left[\mathbf{C}^{-1}\int_{V\backslash\Omega}\mathbf{C}\nabla(\mathbf{u}-\mathbf{u}_0)\,\mathrm{d}\mathbf{x} + |\Omega|\mathbf{E}^\star\right]\\ &= \frac{1}{|V|}\left[\mathbf{C}^{-1}\int_{\partial(V\backslash\Omega)}((\mathbf{C}\nabla(\mathbf{u}-\mathbf{u}_0))\mathbf{n})\otimes\mathbf{x}\,\mathrm{d}\mathbf{x} + |\Omega|\mathbf{E}^\star\right]\\ &= \frac{1}{|V|}\left[\mathbf{C}^{-1}\int_{\partial\Omega^+}(((\mathbf{R}^{-1}-\mathbf{C})\mathbf{F}^\star)\mathbf{n})\otimes\mathbf{x}\,\mathrm{d}\mathbf{x} + |\Omega|\mathbf{E}^\star\right]\\ &= f(\mathbf{C}^{-1}\mathbf{R}^{-1})\mathbf{F}^\star = f\mathbf{S}^{-1}\mathbf{E}^\star, \end{aligned} \tag{33}$$

where $f = |\Omega|/|V|$ is the volume fraction. In the above calculations, the last equality follows from Eq. (29). For the third equality, we write

$$(\mathbf{C}\nabla(\mathbf{u}-\mathbf{u}_0))\mathbf{n} = ((\mathbf{C}-\mathbf{R}^{-1})\mathbf{F}^\star)\mathbf{n} \quad \forall\mathbf{x}\in\partial\Omega^+. \tag{34}$$

This is reasonable since $\mathbf{u}-\mathbf{u}_0$ satisfies

$$\begin{cases} \operatorname{div}[\mathbf{C}\nabla(\mathbf{u}(\mathbf{x})-\mathbf{u}_0(\mathbf{x}))] = 0, & \mathbf{x}\in V\backslash\Omega,\\ \mathbf{u}(\mathbf{x})-\mathbf{u}_0(\mathbf{x}) = \mathbf{F}^\star\mathbf{x}, & \mathbf{x}\in\Omega,\\ (\mathbf{C}\nabla(\mathbf{u}-\mathbf{u}_0))\mathbf{n}(\mathbf{x}) = 0, & \mathbf{x}\in\partial V. \end{cases} \tag{35}$$

Since the volume fraction $f \ll 1$, one may argue that the surface traction on $\partial\Omega^+$ would not be affected by replacing V by $\mathbb{R}^3$ in Eq. (35), i.e., the surface traction on $\partial\Omega^+$ determined by Eq. (35) is the same as that determined by the elasticity problem in Eq. (18) with $\mathbf{F}^0$ replaced by $\mathbf{F}^\star$. Therefore, Eq. (34) follows from Eq. (52) in the Appendix.

To get the total strain, we add the quantity $(1/|V|)\int_V \mathbf{E}[\mathbf{u}_0]\,\mathrm{d}\mathbf{x}$ to $f\mathbf{S}^{-1}\mathbf{E}^\star$. The first of these is independent of the strain in the inclusion, while the second is independent of the applied loads.

We next show that for ellipsoidal inclusions, the loading device energy from an external stress can be written in the form of (14) with $\mathbf{S}_e$ constant. We assume dead loading, so the surface traction $\mathbf{t}(\mathbf{x})$ is independent of the deformation of the specimen. For a displacement

$\mathbf{u}(\mathbf{x})$, the potential energy due to this loading device is

$$\mathscr{E}_{\mathrm{p}}(\mathbf{t},\mathbf{u}) = -\int_{\partial V} \mathbf{t}(\mathbf{x})\cdot\mathbf{u}(\mathbf{x})\,\mathrm{d}\mathbf{x}. \tag{36}$$

In fact any admissible displacement field $\mathbf{u}$ necessarily satisfies Eq. (31). Since only the potential energy difference between displacements is important, we may again choose the reference displacement $\mathbf{u}_0$ as defined by Eq. (32). The potential energy difference between $\mathbf{u}$ and $\mathbf{u}_0$ can be written as

$$\begin{aligned}\mathscr{E}_{\mathrm{p}}(\mathbf{t},\mathbf{u}-\mathbf{u}_0) &= -\int_{\partial V} \mathbf{t}(\mathbf{x})\cdot(\mathbf{u}(\mathbf{x})-\mathbf{u}_0(\mathbf{x}))\,\mathrm{d}\mathbf{x} = -\int_{\partial V}((\mathbf{C}\nabla\mathbf{u}_0)\mathbf{n})\cdot(\mathbf{u}-\mathbf{u}_0)\,\mathrm{d}\mathbf{x}\\ &= -\int_{\partial\Omega^+}((\mathbf{C}\nabla\mathbf{u}_0)\mathbf{n})\cdot(\mathbf{u}-\mathbf{u}_0)\,\mathrm{d}\mathbf{x} - \int_{V\setminus\Omega}\mathbf{C}\nabla\mathbf{u}_0\cdot\nabla(\mathbf{u}-\mathbf{u}_0)\,\mathrm{d}\mathbf{x},\end{aligned}$$

where the divergence theorem has been used. Further, for the given $\mathbf{u}_0$, we define a (symmetric) stress $\mathbf{S}_{\mathrm{e}}:\Omega\mapsto\mathbb{R}^{3\times3}_{\mathrm{sym}}$ such that[2]

$$\begin{cases}\mathrm{div}\mathbf{S}_{\mathrm{e}}(\mathbf{x}) = 0, & \mathbf{x}\in\Omega,\\ (\mathbf{S}_{\mathrm{e}}(\mathbf{x}) - \mathbf{C}\nabla\mathbf{u}_0)\mathbf{n}(\mathbf{x}) = 0, & \mathbf{x}\in\partial\Omega^+,\end{cases} \tag{37}$$

Then, by the divergence theorem, we have

$$\begin{aligned}\int_{\partial\Omega^+}((\mathbf{C}\nabla\mathbf{u}_0)\mathbf{n})\cdot(\mathbf{u}-\mathbf{u}_0)\,\mathrm{d}\mathbf{x} &= \int_{\Omega}\mathbf{S}_{\mathrm{e}}(\mathbf{x})\cdot\mathbf{E}[\mathbf{u}](\mathbf{x})\,\mathrm{d}\mathbf{x},\\ \int_{V\setminus\Omega}\mathbf{C}\nabla\mathbf{u}_0\cdot\nabla(\mathbf{u}-\mathbf{u}_0)\,\mathrm{d}\mathbf{x} &= \int_{\partial(V\setminus\Omega)}((\mathbf{C}\nabla(\mathbf{u}-\mathbf{u}_0))\mathbf{n})\cdot\mathbf{u}_0\,\mathrm{d}\mathbf{x} = 0.\end{aligned}$$

Thus,

$$\mathscr{E}_{\mathrm{p}}(\mathbf{t},\mathbf{u}-\mathbf{u}_0) = -\int_{\Omega}\mathbf{S}_{\mathrm{e}}(\mathbf{x})\cdot\mathbf{E}[\mathbf{u}](\mathbf{x})\,\mathrm{d}\mathbf{x}. \tag{38}$$

In particular, if Ω is an ellipsoidal region and

$$\mathbf{t}(\mathbf{x}) = \mathbf{S}_{\mathrm{a}}\mathbf{n}(\mathbf{x}) \quad \mathbf{x}\in\partial V, \tag{39}$$

where $\mathbf{S}_{\mathrm{a}}\in\mathbb{R}^{3\times3}_{\mathrm{sym}}$, then from Eqs. (32) and the identity $\mathbf{CRCS}^{-1}=\mathbf{C}$ (see Eq. (56)), we know that $\mathbf{u}_0' = \mathbf{u}_0 - \mathbf{RCE}^*\mathbf{x}$ satisfies

$$\begin{cases}\mathrm{div}[\mathbf{C}\nabla\mathbf{u}_0'(\mathbf{x})] = 0, & \mathbf{x}\in V\setminus\Omega,\\ \mathbf{u}_0'(\mathbf{x}) = -\mathbf{RCE}^*\mathbf{x}, & \forall\mathbf{x}\in\Omega,\\ (\mathbf{C}\nabla\mathbf{u}_0'(\mathbf{x}))\mathbf{n}(\mathbf{x}) = 0, & \mathbf{x}\in\partial V,\end{cases} \tag{40}$$

where $\mathbf{E}^* = \mathbf{S}^{-1}\mathbf{C}^{-1}\mathbf{S}_{\mathrm{a}}$. As before, we argue that in the dilute limit, the surface traction on $\partial\Omega^+$ determined by Eq. (40) is the same as that determined by the elasticity equation in (18) with $\mathbf{F}^0$ replaced by $-\mathbf{RCE}^*$. Therefore, from Eq. (52) we have

$$(\mathbf{C}\nabla\mathbf{u}_0'(\mathbf{x}))\mathbf{n} = -(\mathbf{CRCE}^* - \mathbf{CE}^*)\mathbf{n} \Rightarrow (\mathbf{C}\nabla\mathbf{u}_0(\mathbf{x}))\mathbf{n} = (\mathbf{CE}^*)\mathbf{n} \quad \forall\mathbf{x}\in\partial\Omega^+.$$

[2]This can always be done, by for example solving an elasticity problem. Since the modulus for this elasticity problem can be an arbitrary positive-definite tensor, the stress so defined is not unique, but makes no difference for the integral (38).

From definition (37) of $\mathbf{S}_e$, we may choose

$$\mathbf{S}_e(\mathbf{x}) = \mathbf{C}\mathbf{S}^{-1}\mathbf{C}^{-1}\mathbf{S}_a, \quad \forall \mathbf{x} \in \Omega, \tag{41}$$

which indeed has the form (14).

In summary, under the application of surface traction of the form (39), the effective applied stress $\mathbf{S}_e$ in the free energy of the composite system (cf. Eq. (15)) is given by Eq. (41). The average composite strain is $(1/|V|)\int_V \mathbf{E}[\mathbf{u}_0]\,d\mathbf{x} + f\mathbf{S}^{-1}\mathbf{E}^\star$ and $\mathbf{u}_0$ is independent of $\mathbf{E}^\star$. The strain inside the magnetostrictive particles $\mathbf{E}^\star$ is constant and is governed by the minimization problem (27).

5. Application to Ni_2MnGa and terfenol-D composites

We consider two different magnetostrictive materials: Ni_2MnGa and Terfenol-D. The energy-wells of Ni_2MnGa are given by

$$\mathbf{E}_1 = \begin{bmatrix} -0.048 & 0 & 0 \\ 0 & 0.013 & 0 \\ 0 & 0 & 0.013 \end{bmatrix} \pm \mathbf{m}_1 = \pm m_s(1,0,0),$$

$$\mathbf{E}_2 = \begin{bmatrix} 0.013 & 0 & 0 \\ 0 & -0.048 & 0 \\ 0 & 0 & 0.013 \end{bmatrix} \pm \mathbf{m}_2 = \pm m_s(0,1,0),$$

$$\mathbf{E}_3 = \begin{bmatrix} 0.013 & 0 & 0 \\ 0 & 0.013 & 0 \\ 0 & 0 & -0.048 \end{bmatrix} \pm \mathbf{m}_3 = \pm m_s(0,0,1),$$

where strains are measured with respect to the undistorted austenite, and the cubic axes of the austenite crystal is taken as the coordinate system. Also, $m_s = 602\,\text{emu/cm}^3$ is the saturation magnetization of Ni_2MnGa, see Tickle and James (1999). Note that the constant $\gamma = 4\pi$ in *cgs* system of units in which we work here, see Jackson (1999).

With respect to the undistorted crystalline lattice, the energy wells for Terfenol-D are given by the following formulae (no summation)

$$(\mathbf{E}_i, \pm\mathbf{m}_i) = \left(\tfrac{3}{2}\lambda_{111}(\mathbf{e}_i \otimes \mathbf{e}_i - \tfrac{1}{3}\mathbf{1}),\ \pm m_s'\mathbf{e}_i\right) \quad i = 1,\ldots,4,$$

where $\mathbf{1} \in \mathbb{R}^{3\times 3}$ is the identity matrix and the unit vector $\mathbf{e}_i$ can be chosen from

$$\mathbf{e}_1 = \tfrac{1}{\sqrt{3}}(1,1,1), \quad \mathbf{e}_2 = \tfrac{1}{\sqrt{3}}(-1,1,1), \quad \mathbf{e}_3 = \tfrac{1}{\sqrt{3}}(1,-1,1), \quad \text{and} \quad \mathbf{e}_4 = \tfrac{1}{\sqrt{3}}(1,1,-1).$$

Also, $\lambda_{111} = 2\times 10^{-3}$, $m_s' = 800\,\text{emu/cm}^3$, see DeSimone and James (2002).

One can easily check that the energy-wells of Ni_2MnGa and Terfenol-D satisfy the pairwise compatibility conditions (7). At the same time, Ni_2MnGa and apparently also Terfenol-D have high anisotropy energy, and both clearly have mobile interfaces. Thus, it is safe to use the constrained theory to predict their behaviors even though the mechanism of producing strains is different for Ni_2MnGa and Terfenol-D, i.e., twin boundary motion versus domain wall motion.

We assume that the elasticity tensor $\mathbf{C}$ of the matrix is isotropic and so can be described by a Poisson's ratio v and a Young's modulus E. In this case, the demagnetization matrix

D and the Eshelby tensor **S** are well known and general formulae are presented in the Appendix. See also Kellogg (1929) and Mura (1987).

We assume that the direction of the applied magnetic field is along one of the easy axes of the crystal, say, $\mathbf{e}_1$, and that the applied stress $\mathbf{S}_a$ is uniaxial stress along the same direction. (Note $\mathbf{e}_1 = (1\,0\,0)$ for Ni_2MnGa and $(1\,1\,1)$ for Terfenol-D.) Therefore, we have

$$\mathbf{h}_e = h_e\mathbf{e}_1 \quad \text{and} \quad \mathbf{S}_a = S_a\mathbf{e}_1 \otimes \mathbf{e}_1,$$

where h_e and S_a are the magnitude of the applied magnetic field and stress, respectively. We will also assume that the inclusion is an axisymmetric ellipsoid, with the axis of rotation along with the same easy axis $\mathbf{e}_1$.

In the following, we consider the composite strain along $\mathbf{e}_1$. We imagine varying the applied magnetic field h_e at fixed applied stress S_a. In this case, we are interested in the change in strain owing to the applied magnetic field. The *composite strain* is defined by

$$\varepsilon_{\mathbf{e}_1} = f\mathbf{e}_1 \cdot \mathbf{S}^{-1}(\mathbf{E}^\star - \mathbf{E}_0^\star)\mathbf{e}_1, \tag{42}$$

where f is the volume fraction of the magnetostrictive material, $\mathbf{E}^\star$ is the minimizer of problem (27) corresponding to a certain h_e, and $\mathbf{E}_0^\star$ is the minimizer corresponding to $h_e = 0$. With reference to the discussion of Section 4 we note that (42) is also equal to the total strain at h_e minus its value at $h_e = 0$; that is, it is independent of the reference strain introduced in Eq. (33). This is the strain of interest in applications.

For any given volume fraction f, Young's modulus E, Poisson's ratio v, applied stress S_a, applied magnetic field h_e, and shape of the axisymmetric ellipsoid, we calculate the matrix **A** and vector **b** from Eq. (28). The quadratic programming problem (27) can be easily solved numerically, and by Eq. (42) the composite strain can be calculated. Therefore, one can formally write the composite strain $\varepsilon_{\mathbf{e}_1}$ as a function of f, E, v, e, S_a and h_e,

$$\varepsilon_{\mathbf{e}_1} = \varepsilon_{\mathbf{e}_1}(f, E, v, e, S_a, h_e), \tag{43}$$

where $e = b/a$ is the aspect ratio the axisymmetric ellipsoidal inclusion, a is the half-length along $\mathbf{e}_1$ and b is the half length transverse to $\mathbf{e}_1$. In the following, we investigate the behavior of Eq. (43) as a function of all the parameters except the volume fraction f. Since the composite strain is proportional to the volume fraction, we set it to be 0.1 in all calculations.

Fig. 3 shows plots of composite strain versus the applied magnetic field h_e for both Ni_2MnGa and Terfenol-D. In both cases, $E = 0.1$ GPa, $v = 0.4$, $e = 1$, and $S_a = 0$. The "$*$" curves are computed for the minimization problem (16) and the "+" curve is computed for the minimization problem that defines the lower bound in Eq. (30); see also the remarks at the end of Section 3. The curves are symmetric about $h_e = 0$ since the preferred magnetization can be either positive or negative. Consider $h_e > 0$. For small h_e, the composite strain stays at zero. In this region variants with magnetization directed at $-\mathbf{e}_1$ vanish at the expense of those with magnetization directed at $+\mathbf{e}_1$, with no change in macroscopic strain. Once these antiparallel variants are removed, the other disfavored variants such as $(\mathbf{E}_2, \pm\mathbf{m}_2)$ are gradually removed at the expense of the variant $(\mathbf{E}_1, \mathbf{m}_1)$, and so the composite strain changes linearly with the field. This continues until saturation, when the magnetostrictive particles are all the variant $(\mathbf{E}_1, \mathbf{m}_1)$. The absolute value of the largest composite strain is around 9.1×10^{-3} for Ni_2MnGa and 1.4×10^{-3} for Terfenol-D. Note that the composite strain for Ni_2MnGa is negative since the variant $(\mathbf{E}_1, \mathbf{m}_1)$ has a

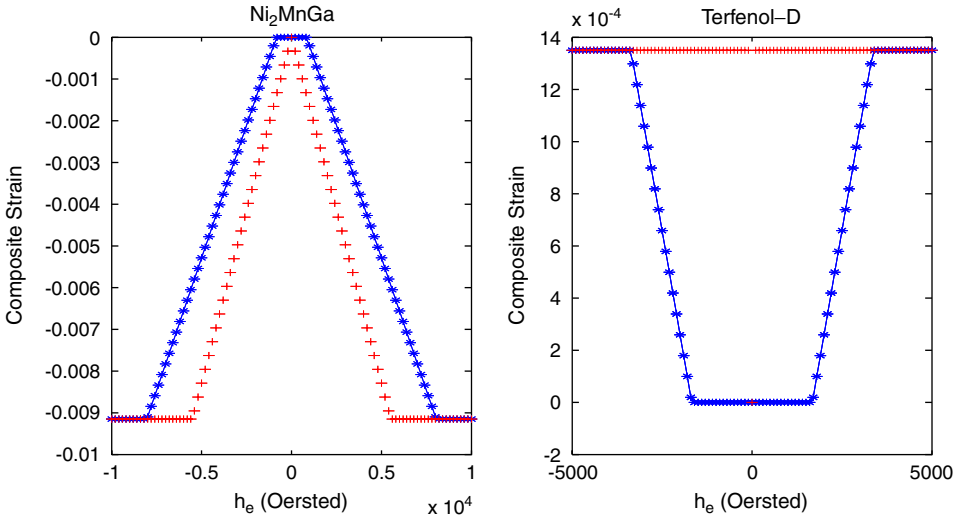

Fig. 3. Composite strain versus applied magnetic field h_e. The curves are computed for Ni_2MnGa and Terfenol-D with $f = 0.1$, $e = 1$, $E = 0.1$ GPa, $v = 0.4$, and $S_a = 0$. The "$*$" curves are computed for the minimization problem (16) and the "+" curve is computed for the minimization problem that defines the lower bound in Eq. (30). Note that on the scale of the Terfenol-D curve, the "+" curve appears to collapse to a horizontal line at the saturation strain with a single point at $h_e = 0$ and zero strain.

negative transformation strain -0.048 associated with $\mathbf{e}_1 = (1\,0\,0)$. In contrast in Terfenol-D the preferred variant for $\mathbf{e}_1 = (1\,1\,1)$ has a positive transformation strain along $\mathbf{e}_1$ and so the composite strain is also positive.

From Fig. 3, the largest value of strain achieved for Ni_2MnGa is about -9.1×10^{-3}. It is interesting to compare this to the strains that can be achieved by some simple composite geometries. For example, consider a lamellar composite where the magnetoelastic particle is a plate with normal along the direction of applied field. In this case the maximum composite strain, assuming no initial bias, is the volume fraction times the variant strain, i.e., -4.8×10^{-3}, with absolute value smaller than -9.1×10^{-3}. Of course, higher strains can be achieved by aligning the long axes of plate- or needle-like particles along the applied field; in this case the full strain -4.8×10^{-2} of the particle can be achieved by the composite if the applied field is large enough.

We consider next the effect of the aspect ratio e on composite behavior. Note that because a is the half length of the axis of rotation, $\log(e) > 0$ corresponds to a disk-like ellipsoid (an oblate ellipsoid), $\log(e) = 0$ is a sphere and $\log(e) < 0$ corresponds to a needle-like ellipsoid (a prolate ellipsoid). The parameters $E = 0.1$ GPa, $v = 0.4$, $h_e = 4000$ Oe, and $S_a = 0$. Fig. 4 shows the composite strain plotted against the logarithm of the aspect ratio e for Ni_2MnGa. We find that prolate ellipsoids and spheres give good composite strain, while oblate spheroids give poor composite strain. The optimal shape for maximum strain is a prolate ellipsoid with $e \approx 0.8$. We note that the specific values depend on the magnitude of the external fields in an inherently nonlinear way. Also, we note that the composite strain in the dilute limit depends on both the particle strain and the strain of the surrounding matrix as generated through the Eshelby tensor. Thus, while isolated infinite rods have the largest strain for a given applied field, they are not effective at generating

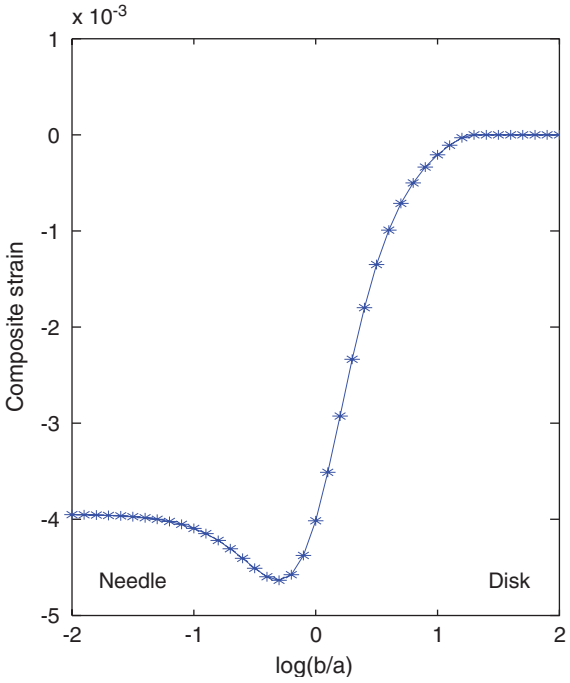

Fig. 4. Composite strain versus log of the aspect ratio. Curve is computed for Ni_2MnGa with $f = 0.1$, $E = 0.1\,\text{GPa}$, $v = 0.4$, $h_e = 4000\,\text{Oe}$ and $S_a = 0$.

strain in the matrix. For all cases we have considered, the strain is fairly insensitive to the shape for all rod-shaped inclusions. In contrast, when the inclusion shape becomes disk-like, the strain rapidly approaches zero. This is because for a disk, the Eshelby tensor $\mathbf{S}$ makes A_{ij} in (28) extremely large, and so the energy of deforming the inclusion becomes prohibitively large.

We consider next the effects of a tensile applied stress S_a for Ni_2MnGa. We choose a tensile stress so that in the absence of a magnetic field, variants $(\mathbf{E}_2, \pm\mathbf{m}_2)$ and $(\mathbf{E}_3, \pm\mathbf{m}_3)$ are favored over variants $(\mathbf{E}_1, \pm\mathbf{m}_1)$. A magnetic field $h_e = 4000\,\text{Oe}$ is then applied, which favors variants $(\mathbf{E}_1, \mathbf{m}_1)$ over the others. Three different shapes of inclusion are considered, corresponding to the aspect ratio $e = 0.01, 1, 100$. Fig. 5 shows that if the applied stress is too small (approximately $<0.05\,\text{MPa}$), it has little effect on the final strain at $h_e = 4000\,\text{Oe}$. On the other hand, when the applied stress gets too large (approximately $>0.1\,\text{MPa}$ for needle-like inclusion, $>0.5\,\text{MPa}$ for disk-like inclusion, and $>11\,\text{MPa}$ for spherical inclusion), the applied stress prohibits any deformation of the composite. At intermediate values the applied stress can lead to larger strain for disk-like and spherical inclusions. There exists some "optimal" value where we have a peak (absolute value) of strain. These conclusions hold for different magnetic field strengths, though the whole plot is shifted to the right (left) for larger (smaller) applied magnetic field.

We next consider the role of the elastic modulus of the matrix. Fig. 6 shows plots of strain versus the logarithm of Young's modulus for both Ni_2MnGa and Terfenol-D. The change of Young's modulus has little effect on the composite strain below a certain value

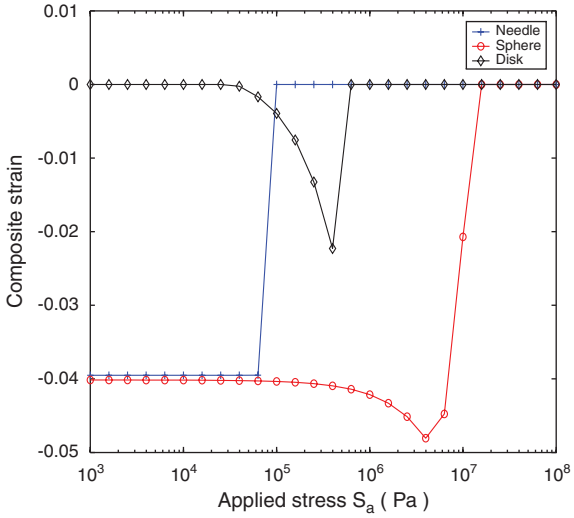

Fig. 5. Composite strain versus the applied stress S_a. Three curves, corresponding to $e = 0.01$, 1 and 100, are computed for Ni_2MnGa. Other parameters $f = 0.1$, $E = 0.1$ GPa, $v = 0.4$, and $h_e = 4000$ Oe.

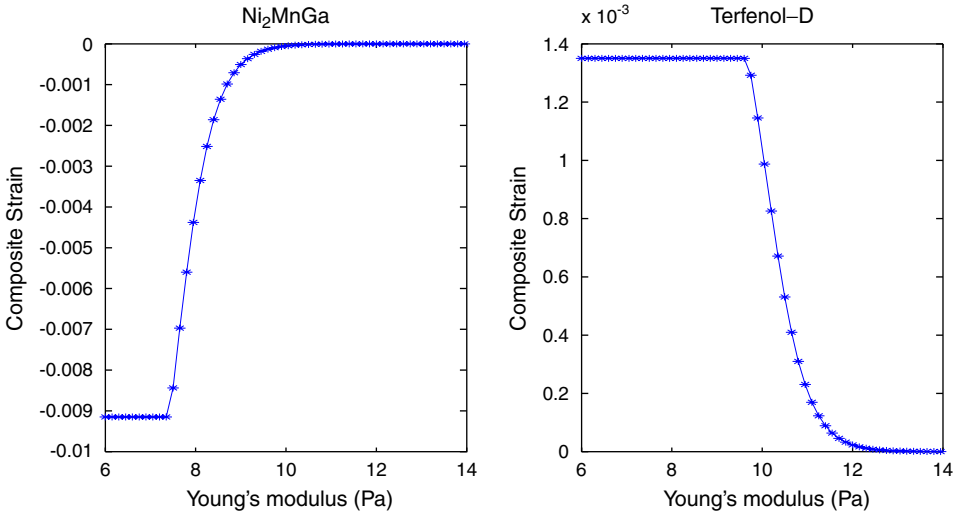

Fig. 6. Composite strain versus the Young's modulus of the matrix. Curves are computed with $f = 0.1$, $e = 1$, $v = 0.4$, $h_e = 4000$ Oe, and $S_a = 0$.

of about 50 MPa for Ni_2MnGa and about 5 GPa for Terfenol-D. This low modulus regime corresponds to where the elastic energy contribution is negligible compared to the magneto-static energy. Once the Young's modulus is large enough, the composite strain drops rapidly to zero, at which point the constraint of the matrix completely prevents deformation of the magnetostrictive particles under the fixed magnetic field $h_e = 4000$ Oe.

For the example shown in Fig. 6, this completely constrained case occurs when the modulus reaches about 1 GPa for Ni_2MnGa and about 100 GPa for Terfenol-D. Finally, we note that the Poisson's ratio of the matrix has little effect on the elastic energy of the matrix, and hence little effect on the magneto-elastic properties of the composite.

The results from Fig. 6 show that in order to obtain a composite with good magneto-elastic properties, the Young's modulus of the matrix cannot be too high. On the other hand, if the Young's modulus is too low, the composite would not be effective in force actuation. One possible criterion for force actuation is the elastic energy stored in the matrix. Fig. 7 shows the elastic energy versus the Young's modulus of the matrix for differently shaped Ni_2MnGa particles. We see that there is a clear maximum in the elastic energy. For values of modulus below this maximizer, the composite strain is essentially saturated and so the elastic energy increases with increasing Young's modulus. However when the matrix becomes too stiff, the composite strain drops as in Fig. 6, and hence so does the elastic energy. The value of the maximum stored energy, and the corresponding Young's modulus, depends on the shape of the inclusion: for spherical and needle-like inclusions the "optimal" modulus is about 0.1 GPa, while for disk-like inclusions, the "optimal" modulus is about two orders of magnitude lower. This result for disk-like inclusions arises because the Eshelby tensor for oblate ellipsoids gets very large. The values of the maximum energy are consistent with Fig. 4: at a Young's modulus of 0.1 GPa, disk-like inclusions have the lowest value of strain, while spheres and needle-like spheroids have similar, higher, values of strain. The corresponding situation for composite with spherical Terfenol-D inclusions is shown in Fig. 8, and compared with Ni_2MnGa. We see very similar behavior to the Ni_2MnGa case, except that the value of the optimal modulus is around two orders higher. This is consistent with Fig. 6 and the fact that the strains in the energy-wells of Terfenol-D are much smaller than in Ni_2MnGa.

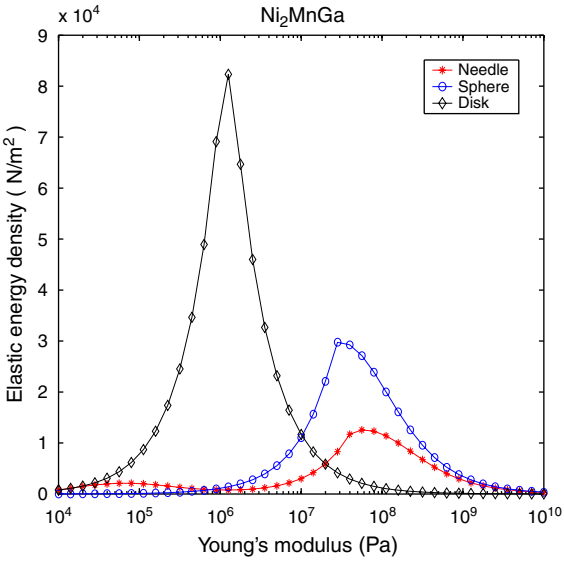

Fig. 7. Elastic energy in the matrix versus the Young's modulus. Three curves, corresponding to $e = 0.01$, 1 and 100, are computed for Ni_2MnGa. Other parameters $f = 0.1$, $v = 0.4$, $h_e = 4000$ Oe, and $S_a = 0$.

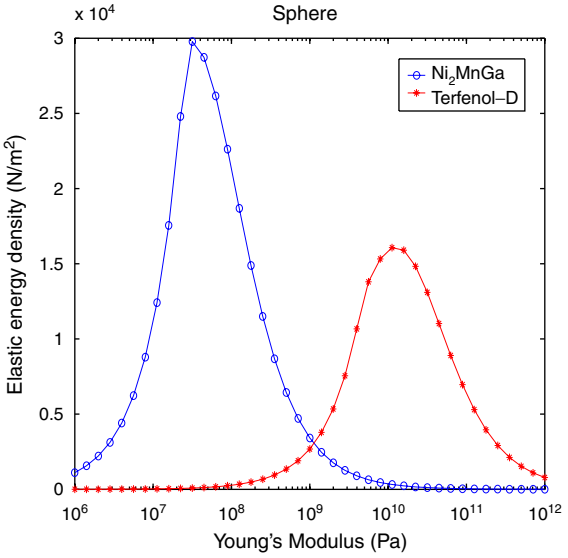

Fig. 8. Elastic energy in the matrix versus the Young's modulus. Two curves, corresponding to Ni_2MnGa and Terfenol-D, are computed with $f = 0.1$, $e = 1$, $\nu = 0.4$, $h_e = 4000\,\text{Oe}$, and $S_a = 0$.

6. Summary

We have considered the problem of predicting the properties of magnetostrictive composites in the dilute limit. We show that the basic variational principle for the composite system (cf. Eq. (16)) can be reduced to a quadratic programming problem by using the constrained theory of magnetostriction together with the special properties of ellipsoids in both magnetics and elasticity. It should be noted that even with these simplifications, the theory we present is nonlinear and nonlocal.

The solutions to the quadratic programming problem yield valuable information about the design of a magnetostrictive composite. In practice, the parameters we can control are the elastic modulus of the matrix, the shape (at least to some degree) of the FSM particles, and the directions and magnitudes of the applied magnetic field and mechanical load. Our calculations suggest one should choose a matrix with an elastic modulus of about 0.1 GPa for Ni_2MnGa particles, and around 10 GPa for Terfenol-D particles. Moreover, one should use spherical or rod-like particles, where the axis of rotation is one of the easy axes of the crystal. The magnetic field should be applied along this same direction, while a biasing stress should be aligned so that it disfavors the variants favored by the magnetic field.

Appendix A. Special properties of ellipsoids

In this appendix, we show that the linear elasticity problem defined in Eq. (18) can be regarded as a type Eshelby inclusion problem. Eshelby (1957) considered the following linear elasticity problem

$$\text{div}[\mathbf{C}(-\nabla\mathbf{u}(\mathbf{x}) + \mathbf{E}^*\chi_\Omega(\mathbf{x}))] = 0, \quad \forall \mathbf{x} \in \mathbb{R}^3. \tag{44}$$

He proved that for an ellipsoidal region Ω, the solution of the above equation is a linear function, i.e., the actual strain $\mathbf{E}[\mathbf{u}]$ is uniform inside Ω. Further the actual strain $\mathbf{E}[\mathbf{u}]$ inside Ω is related to the *eigenstrain* $\mathbf{E}^*$ by

$$\mathbf{E}[\mathbf{u}](\mathbf{x}) = \mathbf{S}\mathbf{E}^*, \quad \forall \mathbf{x} \in \Omega, \tag{45}$$

where $\mathbf{S} \in \mathbb{R}^{3\times3\times3\times3}$ is called the *Eshelby tensor*. The Eshelby tensor depends on the shape of the ellipsoid and the elastic modulus $\mathbf{C}$, and has the following symmetries:

$$S_{ijkl} = S_{jikl} = S_{ijlk}.$$

In fact Eshelby (1957) solved a larger class of inclusion problems, including the inhomogeneous inclusion problem by matching the displacement on $\partial\Omega$ with solution (45).

Walpole (1991) considered the same elasticity problem in Eq. (18) with $\mathbf{F}^0$ replaced by a skew-symmetric $\mathbf{W}^*$. Combining Walpole's solution and Eshelby's solution, one can solve the elasticity problem in Eq. (18) for general $\mathbf{F}^0 \in \mathbb{R}^{3\times3}$.

Using Fourier transformation, we can prove the following general theorem, which covers the magnetostatic problem (2), Eshelby's solution and Walpole's solution. See Liu et al. (2005) for details.

Theorem A.1. *Consider a region $\Omega \subset \mathbb{R}^n$ and the PDE*

$$\operatorname{div}(-\mathbf{L}\nabla\mathbf{u}(\mathbf{x}) + \mathbf{B}\chi_\Omega(\mathbf{x})) = 0 \quad i.e., \; -L_{ipjq}u_{q,ij} + B_{pj}\chi_{\Omega,j} = 0 \quad \forall \mathbf{x} \in \mathbb{R}^n, \tag{46}$$

where $\mathbf{u} : \mathbb{R}^n \mapsto \mathbb{R}^m$ is the unknown field, $\mathbf{B}$ is a constant $m \times n$ matrix, and $\mathbf{L} \in \mathbb{L} \subset \mathbb{R}^{n\times m\times n\times m}$, where

$$\mathbb{L} \equiv \{\mathbf{G} \mid G_{ipjq} = G_{jqip}, G_{ipjq}\zeta_i\zeta_j\eta_p\eta_q \geqslant \alpha\zeta_i\zeta_i\eta_p\eta_p \text{ for } \alpha > 0\}.$$

Also, all derivatives, if necessary, should be understood in weak sense.

If Ω is an ellipsoidal region, then the solution of PDE (46) *restricted to Ω can be expressed as*

$$u_q = R_{ipjq}B_{pj}x_i \quad \forall \mathbf{x} \in \Omega \quad \text{with } R_{ipjq} = \frac{1}{\sigma(n)}\int_{S^{n-1}} \frac{\det(\Lambda)N_{pq}(\hat{\mathbf{k}})\hat{k}_i\hat{k}_j}{|\Lambda\hat{\mathbf{k}}|^n}\,\mathrm{d}\hat{\mathbf{k}}, \tag{47}$$

where S^{n-1} is the surface of the unit ball B_n in $\mathbb{R}^n$ centered at the origin, $\sigma(n)$ is the surface area of this unit ball, $\hat{\mathbf{k}}$ denotes the unit vector in $\mathbb{R}^n$, $\Lambda \in \mathbb{R}^{n\times n}_{\text{sym}}$ is the linear transformation such that $\Omega = \Lambda B_n = \{\mathbf{y} \mid \mathbf{y} = \Lambda\mathbf{x}, \mathbf{x} \in B_n\}$, and $N_{pq}(\hat{\mathbf{k}})$ is the inverse matrix of the matrix $L_{ipjq}\hat{k}_i\hat{k}_j$, i.e., $N_{pr}(\hat{\mathbf{k}})L_{irjq}\hat{k}_i\hat{k}_j = \delta_{pq}$.

It is clear that $N_{pq}(\hat{\mathbf{k}}) = N_{qp}(\hat{\mathbf{k}})$. Thus, the tensor $R_{ipjq} \in \mathbb{R}^{n\times m\times n\times m}$ has the symmetry that $R_{ipjq} = R_{jqip}$. If the coordinate system is chosen to be aligned with the principle axes of the ellipsoid Ω, then $\Lambda = \operatorname{diag}[a_1, a_2, \ldots, a_n]$, where $a_1, a_2, \ldots, a_n$ are the half lengths of axes of the ellipsoid Ω.

The magnetostatic equation (2) with uniform magnetization $\mathbf{m}^0$ on Ω is a special case of (46) with $n = 3$, $m = 1$, $\mathbf{B} = \gamma\mathbf{m}^0$, and $\mathbf{L} = \delta_{ij}$. Therefore, if Ω is an ellipsoid, from the last theorem, the solution inside the ellipsoid can be written as

$$\xi_{\mathbf{m},i}(\mathbf{x}) = D_{ij}m_j^0 \quad \forall \mathbf{x} \in \Omega \quad \text{and} \quad D_{ij} = \frac{\gamma}{4\pi}\int_{S^2} \frac{\det(\Lambda)\hat{k}_i\hat{k}_j}{|\Lambda\hat{\mathbf{k}}|^3}\,\mathrm{d}\hat{\mathbf{k}}.$$

The elasticity problem ($\mathbf{P}^0 \in \mathbb{R}^{3\times 3}$)

$$\operatorname{div}(-\mathbf{C}\nabla \mathbf{u}_{\mathbf{P}}(\mathbf{x}) + \mathbf{P}^0\chi_\Omega) = 0 \tag{48}$$

is also a special case of (46) with $n = 3$, $m = 3$, and $\mathbf{L} = \mathbf{C}$. (The elastic modulus tensor $\mathbf{C}$ has the additional symmetries: $C_{ijkl} = C_{jikl} = C_{ijlk}$.) From Theorem A.1, we have the following formulae:

$$\mathbf{u}_{\mathbf{P}}(\mathbf{x}) = (\mathbf{u}_{\mathbf{P}})_q = R_{ipjq}P^0_{pj}x_i \quad \forall \mathbf{x} \in \Omega \quad \text{and} \quad R_{ipjq} = \frac{1}{4\pi}\int_{S^2} \frac{\det(\Lambda)N_{pq}(\hat{\mathbf{k}})\hat{k}_i\hat{k}_j}{|\Lambda\hat{\mathbf{k}}|^3}\,\mathrm{d}\hat{\mathbf{k}} \tag{49}$$

if Ω is an ellipsoid. From the continuity of surface traction on $\partial\Omega$, we have

$$(\mathbf{C}\nabla \mathbf{u}_{\mathbf{P}}(\mathbf{x}))\mathbf{n} = (\mathbf{C}\mathbf{R}\mathbf{P}^0 - \mathbf{P}^0)\mathbf{n} \quad \forall \mathbf{x} \in \partial\Omega^+. \tag{50}$$

By the divergence theorem, the elastic energy can be written as

$$\int_{\mathbb{R}^3} \nabla \mathbf{u}_{\mathbf{P}} \cdot \mathbf{C}\nabla \mathbf{u}_{\mathbf{P}}\,\mathrm{d}\mathbf{x} = \int_{\mathbb{R}^3} \chi_\Omega \mathbf{P}^0 \cdot \nabla \mathbf{u}_{\mathbf{P}}\,\mathrm{d}\mathbf{x} = |\Omega|\mathbf{P}^0 \cdot \mathbf{R}\mathbf{P}^0 > 0 \quad \forall \mathbf{P}^0 \neq 0. \tag{51}$$

Thus the 4th order tensor $\mathbf{R}$ is invertible and positive-definite in the sense of (51), and so is $\mathbf{R}^{-1}$.

Following Eshelby (1957), if Ω is an ellipsoid, the elasticity problem in Eq. (18) can be solved by matching the boundaries of this problem and problem (48). Since the 4th order tensor $\mathbf{R}$ is invertible, if we choose $\mathbf{P}^0 = \mathbf{R}^{-1}\mathbf{F}$ in problem (48), then the solution $\mathbf{u}_{\mathbf{P}}$ of problem (48) satisfies Eq. (18), i.e., $\mathbf{u}_{\mathbf{P}}$ is exactly the unique solution of the elasticity problem in (18). Thus, from Eq. (50) to (51), the surface traction on $\partial\Omega^+$ and the elastic energy of the elasticity problem in (18) can be expressed as

$$(\mathbf{C}\nabla \mathbf{u}(\mathbf{x}))\mathbf{n} = (\mathbf{C}\mathbf{R}\mathbf{P}^0 - \mathbf{P}^0)\mathbf{n} = (\mathbf{C}\mathbf{F} - \mathbf{R}^{-1}\mathbf{F})\mathbf{n} \quad \forall \mathbf{x} \in \partial\Omega^+, \tag{52}$$

and

$$\frac{1}{2}\int_{\mathbb{R}^3\setminus\Omega} \nabla \mathbf{u} \cdot \mathbf{C}\nabla \mathbf{u}\,\mathrm{d}\mathbf{x} = -\frac{1}{2}\int_{\partial\Omega^+} (\mathbf{n}\otimes\mathbf{u}) \cdot \mathbf{C}\nabla \mathbf{u}\,\mathrm{d}\mathbf{x} = \frac{|\Omega|}{2}\mathbf{F} \cdot (\mathbf{R}^{-1} - \mathbf{C})\mathbf{F}, \tag{53}$$

respectively.

Finally, the classical Eshelby inclusion problem (44) can be regarded as a special case of problem (48) with $\mathbf{P}^0 = \mathbf{C}\mathbf{E}^* \in \mathbb{R}^{3\times 3}_{\mathrm{sym}}$. From the Eshelby solution (45) and Eq. (49), the relation between Eshelby tensor $\mathbf{S}$ and the tensor $\mathbf{R}$ is

$$\mathbf{S}\mathbf{E}^* = \tfrac{1}{2}(\mathbf{R}\mathbf{C}\mathbf{E}^* + (\mathbf{R}\mathbf{C}\mathbf{E}^*)^{\mathrm{T}}), \quad \forall \mathbf{E}^* \in \mathbb{R}^{3\times 3}_{\mathrm{sym}} \quad \text{or} \quad S_{ijkl} = \tfrac{1}{2}(R_{ijmn} + R_{jimn})C_{mnkl}. \tag{54}$$

Therefore, for any $\mathbf{E} \in \mathbb{R}^{3\times 3}_{\mathrm{sym}}$,

$$\tfrac{1}{2}(\mathbf{R}\mathbf{C}\mathbf{S}^{-1}\mathbf{E} + (\mathbf{R}\mathbf{C}\mathbf{S}^{-1}\mathbf{E})^{\mathrm{T}}) = \tfrac{1}{2}(R_{ijmn}C_{mnkl}(\mathbf{S}^{-1})_{klpq}E_{pq} + R_{jimn}C_{mnkl}(\mathbf{S}^{-1})_{klpq}E_{pq}) = E_{ij}.$$

So for any $\mathbf{F} \in \mathbb{R}^{3\times 3}$, let $\mathbf{E} = \frac{1}{2}(\mathbf{F} + \mathbf{F}^{\mathrm{T}})$ and $\mathbf{W} = \mathbf{F} - \mathbf{R}\mathbf{C}\mathbf{S}^{-1}\mathbf{E}$, then

$$\mathbf{W} + \mathbf{W}^{\mathrm{T}} = 0. \tag{55}$$

Further, because of the symmetries of $\mathbf{C}$, we have

$$\begin{aligned}\mathbf{R}\mathbf{C}\mathbf{S}^{-1} &= C_{ijrs}R_{rsmn}C_{mnpq}(\mathbf{S}^{-1})_{pqkl} = \tfrac{1}{2}C_{ijrs}(R_{rsmn} + R_{srmn})C_{mnpq}(\mathbf{S}^{-1})_{pqkl}\\ &= C_{ijrs}S_{rspq}(\mathbf{S}^{-1})_{pqkl} = C_{ijkl} = \mathbf{C}.\end{aligned} \tag{56}$$

Thus, for any $\mathbf{F} \in \mathbb{R}^{3\times 3}$ we have

$$\begin{aligned}\mathbf{F}\cdot\mathbf{R}^{-1}\mathbf{F} &= (\mathbf{W}+\mathbf{RCS}^{-1}\mathbf{E})\cdot\mathbf{R}^{-1}(\mathbf{W}+\mathbf{RCS}^{-1}\mathbf{E})\\ &= (\mathbf{RCS}^{-1}\mathbf{E})\cdot\mathbf{R}^{-1}(\mathbf{RCS}^{-1}\mathbf{E})+2\mathbf{W}\cdot\mathbf{R}^{-1}(\mathbf{RCS}^{-1}\mathbf{E})+\mathbf{W}\cdot\mathbf{R}^{-1}\mathbf{W}\\ &= \mathbf{E}\cdot\mathbf{CS}^{-1}\mathbf{E}+\mathbf{W}\cdot\mathbf{R}^{-1}\mathbf{W},\end{aligned} \tag{57}$$

where in the second equality, we use the symmetry $(\mathbf{R}^{-1})_{ipjq}=(\mathbf{R}^{-1})_{jqip}$, and in the third equality, we use the symmetry $C_{ipjq}=C_{pijq}$ and Eqs. (55)–(56).

In particular, when $\mathbf{C}$ is the isotropic elasticity tensor, i.e., $\mathbf{C}=(\mathbf{C})_{ipjq}=\mu\delta_{ij}\delta_{pq}+\mu\delta_{pj}\delta_{iq}+\lambda\delta_{ip}\delta_{jq}$, then the matrix $N_{pq}(\hat{\mathbf{k}})$ in Eq. (49) is equal to $\frac{1}{\mu}\delta_{pq}-\frac{\mu+\lambda}{\mu(\lambda+2\mu)}\hat{k}_{\mathrm{p}}\hat{k}_q$. From Eq. (54) one has

$$\begin{aligned}R_{ipjq} &= \frac{1}{\mu}\delta_{pq}I_{ij}-\frac{\lambda+\mu}{\mu(\lambda+2\mu)}\mathbb{I}_{ipjq},\\ S_{ipjq} &= \frac{1}{2}(\lambda(R_{iprr}\delta_{jq}+R_{pirr}\delta_{jq})+\mu(R_{ipjq}+R_{ipqj}+R_{pijq}+R_{piqj})\\ &= \frac{\lambda}{\lambda+2\mu}\delta_{jq}I_{ip}+\tfrac{1}{2}(\delta_{pq}I_{ij}+\delta_{pj}I_{iq}+\delta_{iq}I_{pj}+\delta_{ij}I_{pq})-\frac{2(\mu+\lambda)}{\lambda+2\mu}\mathbb{I}_{ipjq},\end{aligned}$$

where

$$I_{ip}=\frac{1}{4\pi}\int_{S^2}\frac{\det(\Lambda)\hat{k}_i\hat{k}_{\mathrm{p}}}{|\Lambda\hat{\mathbf{k}}|^3}\,\mathrm{d}\hat{\mathbf{k}}\quad\text{and}\quad \mathbb{I}_{ipjq}=\frac{1}{4\pi}\int_{S^2}\frac{\det(\Lambda)\hat{k}_i\hat{k}_{\mathrm{p}}\hat{k}_j\hat{k}_q}{|\Lambda\hat{\mathbf{k}}|^3}\,\mathrm{d}\hat{\mathbf{k}}.$$

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