Math 503 **Borel** **9/28/2007**

Here is my effort to provide a complete proof of the Borel Theorem. I gave a really vague outline in class on Friday, October 5, and I am a bit embarrassed. So what's below is part of my apology. I hope it is correct.

Theorem Given any sequence of real numbers, $\{a_n\}_{n\geq 0}$, there is $f \in C^\infty(\mathbb{R})$ with $f^{(n)}(0) = a_n$ for all integers $n \geq 0$.

Proof We are motivated by considering the monomials $\frac{a_n}{n!}x^n$. The values of the derivatives of these at $x = 0$ are correct. We could try to add them up, but the needed convergence of the infinite series makes various changes necessary. So certain "convergence factors" must be introduced. The $n!$ is just a distraction, so I will relabel: $\alpha_n = \frac{a_n}{n!}$. The convergence factors will be constructed from one specific C^∞ bump function, b. Here is what we need about b: it should have support (the closure of the set where it is not 0) equal to $[-1, 1]$; it should be equal to 1 in $\left[-\frac{1}{2}, \frac{1}{2}\right]$, and it should be increasing in $\left[-1, -\frac{1}{2}\right]$ and decreasing in $\left[\frac{1}{2}, 1\right]$. (The last two requirements are not really necessary, but they help me draw pictures in my mind.)

I would like $f(x)$ to be the sum of an infinite series:

$$\sum_{n=0}^{\infty} \alpha_n x^n b\left(\frac{x}{\rho_n}\right) .$$

Here the ρ_n's will be a collection of positive real numbers which will be selected recursively. Each one of them will be modified successively, but there will only be a finite number of modifications for each specific term. Please notice that if the series can be differentiated and evaluated without considerations of convergence, then the equalities $f^{(n)}(0) = a_n$ for all $n \geq 0$ are certainly true.

The C^0 level

Notice that $\alpha_n x^n$ is continuous and its value when $x = 0$ is 0. Therefore we can select ρ_n so that the maximum value of $|\alpha_n x^n|$ on $[-\rho_n, \rho_n]$ is at most $\frac{1}{2^n}$. I'll also ask that ρ_n is itself decreasing as a function of n, and always less than $\frac{1}{2}$ (this will help later in the proof). So we may require: $0 < \rho_{n+1} < \rho_n < \frac{1}{2}$ and $\sup_{x\in[-\rho_n,\rho_n]} |\alpha_n x^n| \leq \frac{1}{2^n}$.

The support of $b\left(\frac{x}{\rho_n}\right)$ is $[-\rho_n, \rho_n]$ (if I scaled this correctly!). Therefore if I am interested in the convergence of $\sum_{n=0}^{\infty} \alpha_n x^n b\left(\frac{x}{\rho_n}\right)$ I can use the implication "absolute convergence implies convergence." I need to consider $\sum_{n=0}^{\infty} |\alpha_n x^n|\, b\left(\frac{x}{\rho_n}\right)$. Since b's values are between 0 and 1, and the support of $b\left(\frac{x}{\rho_n}\right)$ is $[-\rho_n, \rho_n]$, we have

$$|\alpha_n x^n|\, b\left(\frac{x}{\rho_n}\right) \leq \frac{1}{2^n}$$

for all $x \in \mathbb{R}$. Therefore, the Weierstrass M Test applies, and the convergence of the series of functions is absolute for all $x \in \mathbb{R}$ and uniform for all of $\mathbb{R}$.

The C^1 level

If we differentiate the sum we will get

$$\sum_{n=0}^{\infty} n\alpha_n x^{n-1} b\left(\frac{x}{\rho_n}\right) + \alpha_n x^n b'\left(\frac{x}{\rho_n}\right)\left(\frac{1}{\rho_n}\right).$$

If we can prove that *this* series converges uniformly on all of $\mathbb{R}$ we will have shown that the original series converges to a C^1 function and that the derivative of the original series is this series. Notice, please, that shrinking (decreasing) the ρ_n's will still allow the C^0 proof to succeed. I want to just look at an "infinite tail" of the series (the lower summation limit has been changed):

$$\sum_{n=2}^{\infty} n\alpha_n x^{n-1} b\left(\frac{x}{\rho_n}\right) + \alpha_n x^n b'\left(\frac{x}{\rho_n}\right)\left(\frac{1}{\rho_n}\right).$$

We need only prove that this converges absolutely and uniformly, etc. Here let's change the ρ_n's (for $n \geq 2$ only!) by defining $\tilde{\rho}_n = (\rho_n)^2$. Please notice that these numbers also have the properties: $0 < \tilde{\rho}_{n+1} < \tilde{\rho}_n < \frac{1}{2}$.

Since $\tilde{\rho}_n < \rho_n$, the C^0 proof is unaffected. Certainly $\left|n\alpha_n x^{n-1} b\left(\frac{x}{\tilde{\rho}_n}\right)\right| \leq \left|n\alpha_n x^{n-1}\right|$ since the bump's values are between 0 and 1. And the only x's we need consider are in $[-\tilde{\rho}_n, \tilde{\rho}_n]$ since that is the support of $b\left(\frac{x}{\tilde{\rho}_n}\right)$. We know that the sup of $|\alpha_n x^n|$ is at most $\frac{1}{2^n}$ in $[-\rho_n, \rho_n]$.

For $t > 0$, let $V_q(t) = \sup_{|x|\leq t} |Ax^q|$. Then $V_q(t) = |A|\,|t|^q$ and $V_{q-1}(t) = \frac{V_q(t)}{|t|}$ (I'm trying desperately to avoid dividing by 0). Also, $V_q(t^2) = |t|^q V(t)$. This all means that $\sup_{x\in[-\tilde{\rho}_n,\tilde{\rho}_n]} \left|n\alpha_n x^{n-1}\right| = (\rho_n)^{n-2} n \sup_{x\in[-\rho_n,\rho_n]} |\alpha_n x^n| < \frac{n}{2^n}$. So the first part of the sum above has a nice comparison and we can use the Weierstrass M Test.

The second part is more interesting. Look at the general term: $\alpha_n x^n b'\left(\frac{x}{\rho_n}\right)\left(\frac{1}{\rho_n}\right)$. Now b is one specific function and therefore has a specific derivative with compact support. So we can overestimate all the values of $|b'(\text{at any number})|$ by some constant K. This will be a constant multiplier which can be absorbed when we use the Weierstrass theorem. We need to estimate $|\alpha_n x^n|\frac{1}{\rho_n}$ in $[-\rho_n, \rho_n]$. But we have replaced ρ_n by $\tilde{\rho}_n = (\rho_n)^2$. Then we replace the sup of $|\alpha_n x^n|$, which we know is at most $\frac{1}{2^n}$, by $\left(\frac{1}{2^n}\right)(\rho_n)^n$. The "constant" $\frac{1}{\rho_n}$ is changed to $\frac{1}{(\rho_n)^2}$. The effect is thus to replace the estimate of $\left|\alpha_n x^n b'\left(\frac{x}{\rho_n}\right)\left(\frac{1}{\rho_n}\right)\right|$ by the overestimate $K\left(\frac{1}{2^n}\right)(\rho_n)^n\left(\frac{1}{(\rho_n)^2}\right)$.

But we have n at least 2 (what an accident!). So $K\left(\frac{1}{2^n}\right)(\rho_n)^n\left(\frac{1}{(\rho_n)^2}\right) \leq K\left(\frac{1}{2^n}\right)$. So Weierstrass applies to this sum, and we have proved C^1 convergence.

The C^{k+1} level

We proceed inductively. Now the "algebra" may get tedious. We must assume that the sequence $\{\rho_n\}$ has been defined and satisfactorily modified in k previous steps, and that the sum

$$\sum_{n=0}^{\infty} \alpha_n x^n b\left(\frac{x}{\rho_n}\right) .$$

and its first k derivatives all converge "nicely" (uniformly and absolutely, using the Weierstrass M Test). Now look at the $(k+1)^{\mathrm{st}}$ derivative of this series, and let's only consider the tail beginning with $n = k+1$. I hope it looks like this:

$$\sum_{n=k+1}^{\infty} \sum_{j=0}^{k+1} \binom{k+1}{j} \left(\frac{n!}{j!}\right) \alpha_{n-j} x^{n-j} b^{(k-j+1)}\left(\frac{x}{\rho_n}\right)\left(\frac{1}{\rho_n}\right)^{k-j+1} .$$

Of course, the "game" is to overestimate the absolute vlaues of the individual pieces. I hope you can see where the inner sum comes from (Liebniz or product rule for $k+1$ derivatives). Let me first get rid of some "stuff". There are a finite number of derivatives of b involved, and each of them has compact support. I'll overestimate the absolute values of all of them by some number K. Then what we have is the following inner sum:

$$K \sum_{j=0}^{k+1} \binom{k+1}{j} \left(\frac{n!}{j!}\right) \alpha_{n-j} |x|^{n-j} \left(\frac{1}{\rho_n}\right)^{k-j+1} .$$

This is certainly a bit cleaner. And we only need to worry about this for x's in the interval $[-\rho_n, \rho_n]$. And notice also that this is the sum of $k+1$ different pieces, and since n goes from $k+1$ to ∞, we just need to estimate each of the $k+1$ different sums.

Even with all of my good intentions, I'm getting tired. So what happens if we do the same trick and replace ρ_n by $\tilde{\rho}_n = (\rho_n)^2$. Notice that we are only affecting the terms after the $(k+1)^{\mathrm{st}}$, and that, with this sort of scheme, an individual term in the original series such as $\alpha_n x^n b\left(\frac{x}{\rho_n}\right)$ can only have the "original" ρ_n changed at most n times. So we won't run into the problem that I mentioned in class. We will *not* modify the dilation factor of any one b infinitely many times (so the multiplier won't $\to 0^+$!).

Consider $\left(\frac{n!}{j!}\right) \alpha_{n-j} |x|^{n-j} \left(\frac{1}{\rho_n}\right)^{k-j+1}$ for j fixed between 0 and $k+1$, while n "runs" from $k+1$ to ∞. The change from ρ_n to $\tilde{\rho}_n$ will really work. (In fact, a more economical analyst would certainly criticize my approach, so, yet again, apologies!). Since n is at least $k+1$, there are always "enough" powers of x to cancel the growth of $\left(\frac{1}{(\rho_n)^2}\right)^{k-j+1}$. And what about the coefficients $\frac{n!}{j!}$, which, as functions of of n, certainly grow (polynomial in n with degree $n-j$). The induction hypothesis (formulated carefully and correctly!) will counter this with $\frac{1}{2^n}$ as before, and such sums still converges absolutely. In fact, a combination of the two approaches will work for each of the terms, and the induction will be successful. ■ (almost)

This is certainly not a complete proof. But I hope it makes the result more believable. In complex analysis there's the following incredible fact: to prove a corresponding result about $\mathcal{O}(U)$, we will only need to verify the C^0 level! All the other "levels" will automatically be correct. This is fantastic to me, since I am very lazy. It will save us all a great deal of work. The source of this observation is problem 1b in problem set 3.