Math 411 **Homework #2** **9/18/2008**

Due at the beginning of class, Thursday, September 25, 2008*

These problems accompany the text up to but not including the sections on compactness and connectivity.

1. (10 points) Here $\mathbb{N}$ is the set of positive integers, $\{1, 2, 3, \ldots\}$. A positive integer sequence of positive integers (abbreviation here: PISPI) is any function $f\colon \mathbb{N} \to \mathbb{N}$. You may also use traditional sequence notation, $\{a_n\}_{n\in\mathbb{N}}$ with all the a_n's in $\mathbb{N}$.

a) Let $\mathcal{U}$ (for "up") be the collection of all <u>increasing</u> PISPI's. That is, $f \in \mathcal{U}$ if and only if for all n and m in $\mathbb{N}$ with $n \le m$, we have $f(n) \le f(m)$. Prove that $\mathcal{U}$ is uncountable.

Hint Write $\mathcal{U}$ or a subset as an image of an injection from a set which is uncountable.

b) Let $\mathcal{D}$ (for "down") be the collection of all <u>decreasing</u> PISPI's. That is, $f \in \mathcal{D}$ if and only if for all n and m in $\mathbb{N}$ with $n \le m$, we have $f(n) \ge f(m)$. Prove that $\mathcal{D}$ is countable.

Hint Write $\mathcal{D}$ as a countable collection of certain simpler sets, each of which is countable.

2. (8 points) Suppose $d(x, y)$ is a metric on X. Define $D(x, y)$ by $D(x, y) = \min(d(x, y), 1)$.

a) Prove that D is a metric on X.

b) Prove that the d-open sets are the same as the D-open sets. (There are two implications to prove: if $U \subset X$ is open using d, then it is open using D, and if $U \subset X$ is open using D, then it is open using d.)

The *diameter* of a metric space is the sup of the distance between pairs of points. It is an element of $[0, +\infty]$.

c) Prove that any metric space has the same open sets as a metric space of finite diameter.

3. (10 points) This problem considers metrics on the set $\mathbb{R}^2$. If $p_1 = (x_1, y_1)$ and $p_2 = (x_2, y_2)$ are two points in $\mathbb{R}^2$, $d_2(p_1, p_2) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$ is the usual Euclidean metric on $\mathbb{R}^2$. Do <u>either</u> the left-hand side <u>or</u> the right-hand side of what follows.

Define $d_1(p_1, p_2) = |x_1 - x_2| + |y_1 - y_2|$.

a) Prove that d_1 is a metric on $\mathbb{R}^2$.

b) Draw a picture of the d_1 unit ball centered at $(0, 0)$.

c) Find positive numbers C_1 and C_2 so that
$$C_1 d_1(p_1, p_2) \le d_2(p_1, p_2) \le C_2 d_1(p_1, p_2)$$
for all points p_1 and p_2 in $\mathbb{R}^2$.

d) Prove that the d_1-open sets are the same as the d_2-open sets.

Comment d_1 is the *taxicab or L^1 metric.*

Define $d_\infty(p_1, p_2) = \max(|x_1 - x_2|, |y_1 - y_2|)$.

a) Prove that d_∞ is a metric on $\mathbb{R}^2$.

b) Draw a picture of the d_∞ unit ball centered at $(0, 0)$.

c) Find positive numbers C_1 and C_2 so that
$$C_1 d_\infty(p_1, p_2) \le d_2(p_1, p_2) \le C_2 d_\infty(p_1, p_2)$$
for all points p_1 and p_2 in $\mathbb{R}^2$.

d) Prove that the d_∞-open sets are the same as the d_2-open sets.

Comment d_∞ is the *box or L^∞ metric.*

Textbook problems Chapter 2: 6, 7, and 9 parts e) and f). All are worth 10 points.

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* I hope. Let's see how much we cover in class.