(13) 1. Suppose $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ are real sequences, and that for all positive integers, n, $x_n \le y_n \le z_n$. If both $\{x_n\}$ and $\{z_n\}$ converge and have the same limit, L, prove that $\{y_n\}$ converges and its limit is L.

(13) 2. Suppose (X, d) is a metric space. If P and Q are connected subsets of X with $P \cap Q \ne \emptyset$, prove that $P \cup Q$ is connected.

(15) 3. Suppose (X, d) is a metric space.

a) If A and B are subsets of X, prove that $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

b) Give an example to show that the closure of the union of a *countable* number of subsets of X need not be equal to the union of the closures of each of the sets.

c) Give an example to show that $\overline{A \cap B}$ and $\overline{A} \cap \overline{B}$ need not be equal. Here A and B are subsets of X.

(15) 4. Suppose (X, d) is a metric space.

a) If A is a subset of X, prove that $\text{diam}(A) = \text{diam}\left(\overline{A}\right)$.

Comment $\text{diam}(S) = \sup\{d(x, y) : x, y \in S\}$ if $S \subset X$.

b) Give an example of a subset A of X with $\text{diam}(A) \ne \text{diam}(A^{\circ})$ and $A^{\circ} \ne \emptyset$. (A° is the interior of A.)

(15) 5. a) Suppose (X, d) is a metric space, K is a compact subset of X, U is an open subset of X, and $K \subset U$. Prove that there is $r > 0$ so that $\bigcup_{k \in K} N_r(k) \subset U$.

b) Give an example to show that there can be a closed subset C of X and an open subset U of X with $C \subset U$ so that there is <u>no</u> $r > 0$ with $\bigcup_{x \in C} N_r(x) \subset U$.

(14) 6. a) Prove directly from the definition of compactness that the half-open interval $(0, 1] \subset \mathbb{R}$ is not compact. ($\mathbb{R}$ has the usual topology.)

b) Prove that a Cauchy sequence in a metric space is bounded.

(15) 7. Suppose the following is known about three sequences:

If n is a positive integer, then $|x_n - 2| < \dfrac{5}{n}$, $|y_n - 6| < \dfrac{20}{\sqrt{n}}$, and $|z_n - 5| < \dfrac{6}{n^2}$.

Then the sequences $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ converge, and their respective limits are 2, 6, and 5. The sequence whose n^{th} term is $x_n y_n - z_n$ converges and its limit is $2 \cdot 6 - 5 = 7$. Do not prove this, but find and verify a specific n so that $|(x_n y_n - z_n) - 7| < \frac{1}{1{,}000}$. This need not be a "best possible" n but you must supply a specific n and a proof of your estimate.

First Exam for Math 411

October 20, 2008

NAME ______________________________

Do all problems, in any order.

Problem Number	Possible Points	Points Earned:
1	13	
2	13	
3	15	
4	15	
5	15	
6	14	
7	15	
Total Points Earned:		